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Plasma resonance in ionospheric irregularities

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With 16 figures in the text

Summary

The theory of meteor ionisation is reviewed, and the properties of the ionised trails produced by meteors are estimated theoretically. Theories for radio reflections from meteor trails are critically examined, and an accurate scattering theory is formulated. It is found that a narrow trail with high electron density will behave as a resonator and give a strong echo for waves polarised with the electric vector normal to the trail. There is no resonance, and a weaker echo, when the electric vector is parallel to the trail. The effect is due to space charge induced in the boundaries of the trail, as in the plasma resonance studied by Tonks and Langmuir. It is of decisive importance for meteor echoes on wave lengths of some metres.

The predicted polarisation is confirmed in laboratory experiments by Romell.

Polarisation measurements on radio echoes would be a crucial experiment for testing the existence of the predicted resonance in meteor trails. The effect is more important on longer wave lengths, and may cause the observed rise in the rate of meteor echoes between 4 and 8 metres.

The discussion is extended to other scattering centres in the ionosphere, and it is found that any small irregularity can show plasma resonance. It is pointed out that a moderate turbulence in the main ionospheric layers would produce resonators, which can contribute to the pronounced fluctuations observed in single ray fading.

Introduction

In the observations of radio echoes from the upper atmosphere we can recognise echoes from three classes of reflectors with different geometrical forms. The ordinary echoes from the ionospheric layers can be described as reflections from plane stratifications of electrons, the transient echoes associated with meteors are reflections from cylindrical distributions, and a number of fading and diffraction effects are due to isolated scattering centres, which may be idealised as spherical electron distributions.

This point of view comes naturally when we want to study the theory for reflection of radio waves by the various structures. The permissible approximations depend essentially on the geometrical shape and dimensions of the reflectors, and the choice of appropriate coordinates is determined by their symmetry properties. In addition to this mathematical argument for the significance of the geometrical shape, the present work will show that with cylinders or spheres new physical effects come in, which are not important for plane stratifications.

The theory of the common short wave echoes from the ionosphere is fundamentally a ray theory. Its most important elements are Snell's law together with a mathematical analysis of the propagation of plane waves in a homogeneous electron gas. This approximation is sufficient for the slow variations of refractive index in the main ionospheric layers. The theory accounts well for the shape of the various ray paths, the total reflection for frequencies below a critical frequency, and typical phenomena such as the skip zone where the field strength is reduced (Fig. 1).

The theory can be refined in various ways. The geomagnetic field is important because the electron gas becomes double refracting, and an incident beam is split into two rays with different phase velocities and polarisations. The two rays will be bent along different paths in the ionosphere and be reflected at different heights, which explains the observed double splitting of ionospheric echoes.

The physical principles are clear enough to make the theory useful even when conditions deviate from the idealised situation. But since the theory is a ray approximation, the disturbances must be smooth and extended. It explains, however, effects such as modifications due to the earth's curvature, or the M-shaped ray paths which can occur when the lower of two ionised layers contains local patches of increased ionisation (Fig. 2).

For the comparison with the later discussion of reflections from cylinders or spheres it is important to note that a plane transverse wave in a homogeneous electron gas does not induce any space charges. Since the conventional ionospheric theory is built up on the behaviour of a plane wave in a homogeneous region, the effect of space charges is excluded from the beginning, and this limits any attempt to adapt ionospheric theory to the reflection of radio waves by thin cylinders or small spheres.

In recent years systematic studies have been carried out on radio echoes due to localised disturbances in the upper atmosphere. Among a number of puzzling effects two main types of sporadic echoes have been recognised. One is due to ionisation produced by the passage of meteors through the atmosphere, the other appears as fluctuations in the common ionospheric echoes and is due to a kind of localised scattering centres. The location and physical constitution of the ionised trails left behind meteors are fairly well known, but the precise location or the constitution of the second type of scatterers are unknown at present. The work on echoes from meteor trails has been reviewed from the observational side by LOVELL (1948, 1950) and from the theoretical side by HERLOFSON (1948), the work on fading of radio waves has been reviewed by RATCLIFFE (1948).

Judging from the observed intensities, we may expect the size of the ionised regions which cause these transient echoes to be comparable with, or smaller than, the wave lengths used for the observations. Since the usual ionospheric theory is a ray theory, it is not of great help for the discussion of the transient reflections. The purpose of the present paper is therefore to develop *ab initio*

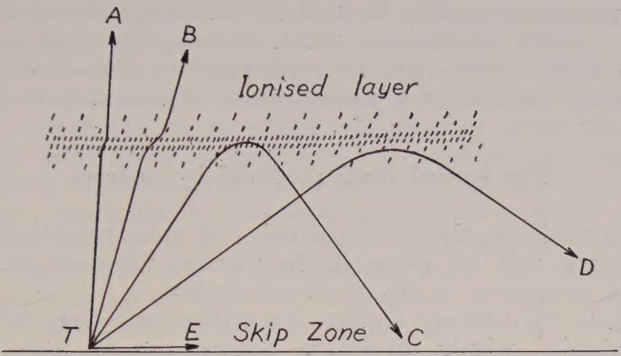

Fig. 1. Ionised layer transmitting rays TA and TB at steep incidence, reflecting rays TC and TD at lower elevation. The skip zone EC is located outside the range E of the attenuated ground wave and inside the range C of the steepest reflected ray.

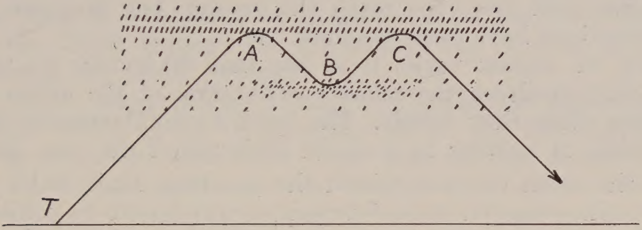

Fig. 2. M-shaped path $TABC$ of ray which penetrates a lower layer, is reflected at A by a high layer, and is bent upwards again at B to a new reflection at C owing to a local increase in ionisation round B.

a theory which can be applied to echoes from meteor trails and diffraction by ionospheric scattering centres. The mathematical discussion must be limited to simple structures with cylindrical or spherical symmetry, but an effort is made to extract the significant physical principles from the formal conclusions. This is necessary in order to apply the results to the actual disturbances in the atmosphere, because their shapes are not likely to be so simple as assumed in the mathematical analysis.

It was noted earlier that the effect of space charge is ignored in the usual ionospheric theory. On the other hand, it will appear later that the incident radio wave will induce space charges in the boundaries of the localised disturbances. This effect is of decisive importance for the echoes, and must always be included.

Under certain conditions the space charges turn the ionised plasma into a resonator. This plasma resonance is an effect known from laboratory experiments on gas discharges (Tonks and Langmuir 1929, Tonks 1931). In the present theory it appears in the form of excessively strong echoes for certain polarisations of the incident radiation, a conclusion which has been verified in recent experiments by Romell (1951).

The term plasma resonance is used for the space charge effect to distinguish it from plasma oscillations, which is now a common term for certain instabilities in ionised gases. Plasma oscillations are phenomena quite different from the resonant response of a plasma to an impressed wave field.

The ionised trails produced by meteors

As a background for the later discussion of radio reflections from meteor trails, a summary will be given of the physical processes which take place when a meteor evaporates in the upper atmosphere.

The views held by different authors are rather conflicting, and the estimates of meteor ionisation widely divergent. Current theories have been critically reviewed by HERLOFSON (1948), and it is concluded that the work of ÖPIK (1933, 1937, 1940) is the most reliable basis for calculations.

It is known from visual and photographic triangulation that the common shooting stars occur at heights between 80 and 120 kilometres, and a typical length of the luminous track is about 30 kilometres. Bright fireballs flare up at lower heights and can even reach the ground, but they are so rare that no radio observations have been obtained yet.

The velocity of impact varies from less than 20 km/sec up to 72 km/sec, and depends very much on the relative orientation of the orbits of the earth and the meteor when they collide. The earth's orbital velocity is 30 km/sec, and if a particle is moving in a closed orbit round the sun, its heliocentric velocity near the earth can not exceed the parabolic limit $30\sqrt{2} = 42$ km/sec. We note that the observed impact velocities correspond to collisions between the earth and meteors moving in elongated elliptical orbits. The lower values occur when a meteor overtakes the earth from the rear in its orbit, while the upper limit corresponds to a head on collision between the front side of the earth and a meteor moving in the opposite direction. There has been considerable controversy about possible heliocentric velocities greater than the parabolic limit, but reliable visual and photographic observations supported by recent radar determinations (ALMOND, DAVIES and LOVELL, 1950) indicate that all meteors observed belong to the solar system.

The chemical composition of meteors is known from the larger bodies which fall down to the ground, and the analyses are in agreement with observed spectra of the light emitted by shooting stars (MILLMAN, 1932). Prominent elements in the meteors are iron, nickel, silicon, and oxygen. A notable feature of the spectra is the predominance of emission lines belonging to levels of low excitation in the meteor elements. This is to be expected if the light is due to collisions between atoms and molecules at common meteor velocities. In collisions of this type it is a general rule that only a minute fraction of the relative kinetic energy is converted into excitation or ionisation, and nearly all the energy is dispersed by elastic scattering (cf. MASSEY, 1949).

At the heights where most meteors evaporate, the mean free path of the air molecules is of the order of centimetres or longer. On the other hand, it will appear as a result of the following discussion that the dimensions of the common meteors are of the order of a millimetre or smaller. This means that the significant process during the passage through the atmosphere is the impact of single air molecules against the front surface of the meteor, while a hydrodynamic flow pattern will not be developed round the body.

If we consider a meteor with a typical velocity of 40 km/sec, it is readily found that the kinetic energy is several hundred electron volts per atom. As seen from the meteor, the air molecules will impinge on the front surface with energies of the same order of magnitude. It is known from experiments on sputtering and on accommodation coefficients that in this energy range most collisions between gas molecules and solid surfaces are inelastic, and the excess energy goes into heating of the solid body. Since the binding energy of the atoms in the meteor is only a few electron volts, the energy given up by each trapped air molecule is sufficient to evaporate a large number of meteor atoms. This is significant, because it means that when a meteor has intercepted an air mass of, say, 1—2 % of its own mass, it has received enough heat to be completely disrupted into separate atoms. The heat is received gradually during the flight, and the meteor atoms are dispersed in a channel along the orbit.

Since the intercepted air mass is negligible compared with the mass of the meteor, the atmosphere does not retard the body, but breaks it up only into single atoms. This is confirmed by observations of the velocity, which show that during the flight through the atmosphere the decrease in speed is only a few per cent of the initial velocity.

A meteor which penetrates to a height of about 100 kilometres, will have swept up about 0.001 grams of air per cm^2 of the front surface. This air mass is able to evaporate about 100 times more of solid matter in the meteor. Hence, if a meteor disappears at that height, its diameter must correspond to about 0.1 grams of solid matter per cm^2 of the front surface, which shows that the dimensions are a few tenths of a millimetre. Similarly, a meteor reaching down to about 80 kilometres above ground level, has dimensions of the order of a millimetre. This shows that the assumption of a large mean free path of the air molecules compared with the dimensions of the meteors is consistent with the conclusions of the physical discussion, and with the observed heights of shooting stars.

We note that the two examples quoted above correspond to solid bodies which contain about 10^{17} and 10^{20} atoms, and it will be shown presently that these limits include most of the visual meteors and the brighter telescopic ones.

An atom which has evaporated from the surface of the meteor, will leave the body with thermal velocity, and therefore retain its cosmic velocity relative to the surrounding air. Since the mean free path of the liberated atom is much longer than the size of the meteor, it is most likely that the first collisions with air molecules after the escape will take place at a distance of a few mean free paths from the meteor orbit. The collisions between free atoms and air molecules are responsible for the light and ionisation produced by the meteor.

The meteor atoms and the air molecules have comparable masses, so the meteor atom will lose most of its kinetic energy after a few elastic collisions, and it is virtually the first collision only which is important for ionisation. This is vital for the whole theory of plasma resonance in meteor trails, because it means that at the moment of creation all the free electrons due to the meteor are located inside a cylindrical distribution with an effective radius of a mean free path, i.e. a distance of the order of centimetres or somewhat larger.

The conclusions so far are fairly safe, and can be put into a quantitative form after a few simple calculations (cf. HERLOFSON, 1948). A much more difficult problem is to estimate the total amount of light and ionisation produced. It is known from experiments (cf. LANDOLT-BÖRNSTEIN, 1950) that the yield of excitation and ionisation by atoms with energies of a few hundred electron volts is extremely small compared with electrons of the same energy, and MASSEY (1949) has given theoretical reasons why the collisions will be nearly adiabatic unless the energy transfer is very small compared with the kinetic energies of the partners. Accurate experimental data are difficult to obtain, because disturbing effects due to the efficient electrons can easily mask the effects due to the atomic collisions. Judging from the meagre data available, it is likely that the energy of an iron atom with a velocity of 40 km/sec will be divided between heat, light and ionisation in the ratios $10^4 : 10^2 : 1$. The light will correspond to transitions between levels of low excitation, and the electrons produced will be slow.

If these data are accepted, the total amount of light can be readily calculated for the meteoric masses deduced above from the depth of penetration into the atmosphere. It is found that the smaller mass would correspond to a telescopic meteor of about 8th magnitude, whereas the larger mass would appear as a bright meteor of 1st magnitude. These figures show that the theoretical values for brightness and heights taken together are related and are in reasonable agreement with the observations. This may be taken as an indication that the theory is not seriously wrong.

The probability for ionisation is of the order of 10^{-2} per atom. The ionisation is spread fairly uniformly over a path of about 30 kilometres. With the same data as before, we can then find the total number of electrons produced by typical meteors. We are, however, more interested in the number of electrons α produced per cm length of the trail. Dividing the total number of electrons with the length of a typical path, we obtain the following figures:

Bright meteor: $\alpha \approx 10^{11} - 10^{12}$ electrons/cm path.
Faint, visual meteor: $\alpha \approx 10^{10}$ electrons/cm path.
Meteor somewhat fainter than the visual limit: $\alpha \approx 10^{9}$ electrons/cm path.

There will, of course, be considerable fluctuations in these figures depending on the meteor's velocity and composition. The accuracy in the final statements has deliberately been limited to the nearest power of ten, and with present data any better agreement between theory and observations would be entirely fortuitous. But the rough estimates of the radius of the ionised trail and the number of electrons per cm path are valuable for the following theory of radio reflections from the trail, because they give an idea of the range of parameters which should be considered in the calculations of echo strength.

It should be mentioned that a certain amount of observational material (SKELLETT 1932, 1935, PIERCE 1941, 1947) has been discussed on the assumption that the major part of the kinetic energy of the meteor appears as ionisation. For the reasons set out in the preceding discussion this view can not be accepted, and a reinterpretation of the observations would be necessary.

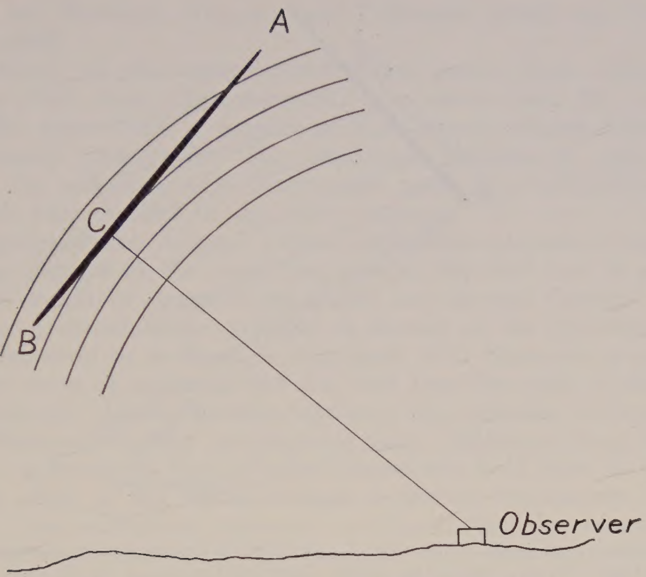

Fig. 3. The greater contribution to an echo comes from the region C where the wave surfaces from the observer O are tangent to the meteor trail AB, and electrons scatter coherently.

Some properties of the ionised trails are immediately recognised in the radio echoes. Most observations of meteors have been carried out on wave lengths between 4 and 10 metres (cf. LOVELL 1948, 1950). On much shorter wave lengths the response from the trails becomes markedly weaker, and on longer wave lengths the echoes which clearly come from meteor trails tend to be masked by a background of scattered echoes from various irregularities in the ionosphere. Further systematic work on the longer wave lengths would be desirable in order to assess the contribution of meteors to the higher rate of scattering.

Since the initial meteor ionisation is confined to a long straight channel with diameter smaller than the wave lengths used for observation, the echo will mainly be observed from a section round the point of shortest distance from the observing station (Fig. 3). The effect is similar to the well defined gleam seen when the sun is shining on a thin metal wire, and the formal analysis can be done by means of Fresnel zones along the trail in the same way as for the reflection of a plane light wave by a perfect mirror (LOVELL and CLEGG, 1948). This aspect sensitivity is apparent in all observations. The Giacobinid shower on October 10th 1946, when meteors could be seen coming in along parallel lines at the rate of several hundred per minute, afforded a particularly striking illustration. With a 4 metre radar equipment pointed in a direction perpendicular to the parallel trails the full rate was observed (LOVELL, BANWELL and CLEGG, 1947), but when the aerial was directed parallel to the stream, the echo rate fell temporarily to 4 per cent of its previous value.

The theoretical analysis of the aspect sensitivity can be carried one step further. The theory of ion production indicated that the electrons are pro-

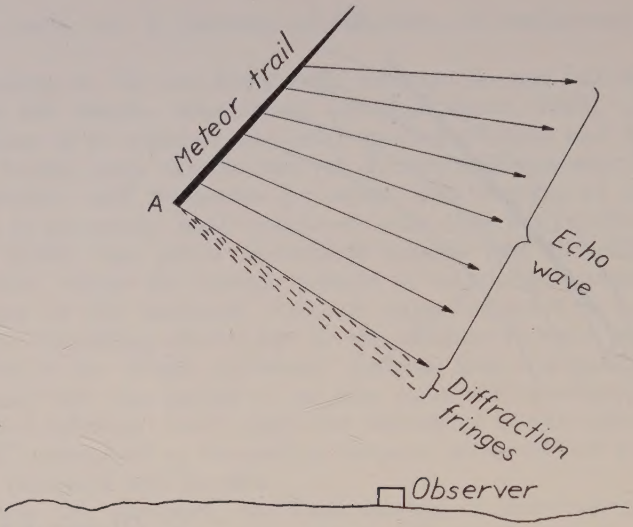

Fig. 4. The ionised trail is fully developed right behind the approaching meteor, and diffraction fringes develop along the sharp sideways limitation of the echo field.

duced during the first few collisions between the newly liberated meteor atoms and air molecules. Hence, the electron density goes up abruptly to its full value right behind the meteor. A reflected wave train on its way back from a meteor trail should therefore be sharply limited sideways owing to the marked rise in reflecting power at the head of the ionised column (Fig. 4). Along the sharp boundary of the wave train diffraction fringes will be formed, and knowing the wave length and the distance to the trail, the separation of the fringes as they sweep across the observing station can be accurately calculated. With distances to the meteor of 100 kilometres or more, and a wave length of 4 metres, the separation of the first few fringes will be of the order of some hundred metres, and with ordinary meteor velocities they will pass over the observer at intervals of the order of 0.01 seconds.

It is clear that if the passage of the fringe system could be followed by means of radio pulses at intervals of a millisecond, and the distance of the trail were also observed, the fringes would allow a velocity determination entirely by radar measurements, and the method could be used by day and night. The experiment has been brilliantly executed by DAVIES and ELLYETT (1949) and developed into a precision method for velocity determination, of particular interest for the study of meteor showers incident on the day side of the earth.

Somewhat different techniques have been used by other workers with equipment particularly suitable for detection of the very faint echoes before the meteor reaches the footpoint of the perpendicular from the observer to the orbit. The first pure radio measurements of meteor velocities were made by HEY, PARSONS and STEWART (1947) who measured the range of the approaching head of the ionised column as function of time. The Doppler effect in the faint echo due to the approaching head has also been used for velocity determinations by MANNING, VILLARD and PETERSON (1949) and by MCKINLEY (cf. LOVELL, 1950).

For the theory of the reflection of radio waves these observations are significant as they show that the main echo comes from the section of the trail where the wave front is parallel to the ionised column, although oblique echoes are faintly observable. The important problem in echo theory will therefore be the reflection of a plane radio wave by a cylindrical column of ionisation with axis parallel to the wave planes.

The occurrence of well defined Fresnel diffraction patterns is consistent with the theoretical picture of the ionisation process, but the test is not crucial as such an effect would be expected on nearly any meteor theory.

There is some observational evidence in favour of the theoretical view that the initial ionisation is confined to a column with diameter comparable with the mean free path of atoms in the air, and that the trail is afterwards dispersed by diffusion. At the heights involved, the diffusion of slow electrons is markedly influenced by the geomagnetic field. Diffusion along the magnetic field lines is unhindered, but diffusion across the field lines is slowed down. The magnetic effect is not strong enough to act on the heavier positive ions, but even so, the ambipolar diffusion coefficient for the ionised column is smaller in a direction normal to the magnetic field than along the field lines. It has been pointed out by HERLOFSON (1948) that this may influence the rate of decay of the meteor echoes, which should become dependent on the angle between the magnetic field lines and the direction of observation. A meteor trail observed near the magnetic zenith would diffuse rapidly in the direction of the line of sight, and destructive interference over the cross section of the trail would soon reduce the echo amplitude. On the other hand, a trail observed on the great circle 90° away from magnetic zenith would diffuse more slowly in the direction of the line of sight, and the coherence between the contributions to the echo from the different electrons would be preserved for a longer time.

Qualitatively, this prediction has been verified in some cases by LOVELL (1948), who directed an aerial system in various azimuths, keeping the elevation constant. During two showers it was found that the average echo duration was 2 to 3 times longer in a northerly direction than in other azimuths. In the north, the line of sight was nearly normal to the magnetic field, while the angle was oblique for the other azimuths. This indicates that diffusion, rather than recombination, determines the rate of decay of the echoes. Diffusion is clearly important for the dispersal of a thin column of ionisation, but less important for a thick column. This evidence — admittedly meagre — is therefore in favour of the theoretical view that the ionised column is thin when it is formed.

Further observations on the connection between echo decay and the geomagnetic field would be desirable, especially in showers with radiation point near the local magnetic zenith, such as the Lyrids for the station near Manchester. In particular, their behaviour should be studied in the light of the more refined reflection theory which is given in the present paper.

The various effects which have been discussed, do occur clearly in many cases, and are important for constructing and checking theoretical models which are not only simple, but also applicable to the geophysical problem.

However, it should be stressed, that in a great number of cases the simple effects are masked by all kinds of regular and irregular fluctuations, which are due to turbulence and large scale wind systems in the upper atmosphere. Direct evidence for such winds is afforded by photographs which show the progressive distortion of luminous trails sometimes left behind meteors for half an hour or more (e.g. STÖRMER, 1937), and the fluctuations in the radio echoes (GREENHOW, 1950) are clearly consistent with deformations of the trails due to the observed winds. The echoes of longer duration are more liable to be disturbed by fluctuations than the short ones with durations of a few tenths of a second. The simple mathematical models which will be used for the reflection theory, are therefore strightly applicable only to the first few tenths of a second, before appreciable distortion has occurred, but the physical principles extracted from the mathematical analysis can be used when discussing the echoes which last for $^1/_2$—1 second or more.

Previous theories of scattering by an ionised cylinder

In connection with geophysical problems several discussions have been given of the scattering of radio waves by straight ionised trails. Most of the solutions proposed are adaptations to the present problem of theories which have been developed for other branches of physics, and only one brief report (FEINSTEIN, 1950) is available on an attempt to start the discussion from fundamental principles.

BLACKETT and LOVELL (1941) have considered the possibility of detecting cosmic ray showers by radio methods, and have, as part of the discussion, calculated the energy scattered by a uniform distribution of free electrons along a straight line. Their result has been used extensively by LOVELL & CLEGG and their collaborators (1948) to calculate the amount of ionisation in meteor trails from the observed echo intensities and the results are in fair agreement with the estimates of electron density deduced from the theory of meteor ionisation.

In the theory it is assumed that each electron scatters according to Thomson's formula, and the variations in the phase of the scattered radiation arriving from different sections of the straight line are taken into account in the same way as in the theory of Fresnel diffraction at a plane surface. From a physical point of view the discussion resembles the calculation of X-ray scattering by atoms. The result is applicable if the electron density is so small that the electrons do not disturb each other. But it is rather difficult from the theory in this form to state exactly when the electrons will begin to interact, or to screen each other from the incident radiation, and the effect of deviations from the idealised situation is hard to assess.

A simple extension of BLACKETT and LOVELL's work to include the phase differences over the cross section of a cylinder with finite diameter has been pointed out by HERLOFSON (1948). Owing to destructive interference the phase differences will reduce the echo amplitude as the trail expands, and to some extent an effect of this kind is confirmed by observations of echo durations (LOVELL, 1948).

The interference effect, and the reasonable agreement between ionisation theory and electron densities deduced via BLACKETT and LOVELL's formula, can be taken as indications that the calculation is a step in the right direction, and the main question is how densely the electrons can be packed before mutual interaction becomes important.

An entirely different solution has been proposed by PIERCE (1938) in an attempt to adapt the usual theory of radio reflections from the ionosphere. It is assumed that inside a certain cylinder surface the electron density is so high that the surface acts as a perfect reflector in the same way as the plane of total reflection in the ray theory of ionospheric echoes. This approximation is valid if the diameter of the cylinder is so large that the evanescent wave which always penetrates some distance into the region of negative dielectric constant, does not reach right into the centre of the cylinder. It is unlikely, however, that an ordinary meteor can produce sufficient ionisation for the theory to be applicable. It will be shown later that for meteor trails with up to 10^{12} electrons per cm path there are two possible cases, depending on the polarisation of the radio wave. The evanescent wave field will either fill the whole cylinder even when the dielectric constant is negative in the core, and BLACKETT and LOVELL's theory is then the correct approximation, or plasma resonance will be important and no modification of a ray theory will be applicable. This means that PIERCE's approximation is certainly correct when applied to cylinders which are large and dense enough, but if the ionisation theory described in the previous section is right, this condition is not fulfilled in meteor trails.

A third solution is suggested by ECKERSLEY (1940) who compares the scattering of radio waves with the deflection of fast electrons in a Coulomb field. The approach is somewhat unfamiliar, but to some extent this view is admissible, because RUTHERFORD's formula is also the wave mechanical solution for scattering of de Broglie waves in a Coulomb field (cf. MOTT & MASSEY, 1949, Ch. III). For one plane of polarisation in the cylindrical case and for one set of modes in the spherical case, the present problem is really reduced to the solution of an ordinary wave equation, but it will appear that ECKERSLEY's analogy can not hold for both polarisations or for both sets of modes at the same time. It would also be hard to justify the strong concentration of electrons at the centre necessary in order to create a distribution of the dielectric constant equivalent to the potential of a Coulomb field.

The scattering due to ionised columns has also been briefly considered by RATCLIFFE (1949).

In a preliminary communication, the results of a rigorous mathematical discussion are reported by FEINSTEIN (1950). The conclusions quoted differ entirely from the results of the present paper and are not supported by ROMELL's experiments (1951), but until the detailed calculations are available, an accurate comparison of the results due to FEINSTEIN and the present writer is difficult.

Formulation of a linear scattering theory

The main difficulty with the previous solutions is to state precisely how far the various approximations are valid. Undoubtedly, several of the calculations are correct and applicable over a certain range of the parameters involved, but each theory in itself does not contain any indication of when it ceases to be valid, or what happens beyond the limits of validity. This defect comes in because each solution is closely linked to a certain branch of physics, and contains a number of tacit assumptions which are correct for that branch, but which may, or may not, be tenable in the present reflection problem. The only way out is to formulate the scattering problem on a basis so general that the whole field can be surveyed by means of one theory, and each of the earlier calculations appears as a limiting case under well defined conditions.

With no ionised column present, a plane wave is a solution of MAXWELL's equations everywhere, and is propagated without deformation. When the ionised column is introduced, the electromagnetic field must still satisfy MAXWELL's equations. But in the region where charges are present, the current due to the motion of the charges must be added to the displacement current, and this will distort the previous plane wave in this region. It will then be impossible to match a pure plane wave outside the column to the distorted field inside the column and still satisfy MAXWELL's equations. The distortion inside the column will, as it were, make itself felt in the wave pattern outside owing to the continuity of the field vectors. This disturbance will fall off with increasing distance from the column, and at large distance there is only left a diverging cylinder wave.

In the geophysical application, the original plane wave is identified with the incident wave due to the radio transmitter, and the diverging cylinder wave with the echo from the meteor trail.

The solutions of scattering problems of a similar kind are well known (cf. WOLFSOHN, Handbuch der Physik). In connection with the early theory of HERTZ' experiments on the polarisation of electromagnetic waves by wire gratings, the disturbance due to a metal wire in a plane radio wave was calculated by SEITZ (1905) and IGNATOWSKY (1905). Similar calculations for dielectric cylinders have been done by SCHÄFER and his collaborators (SCHÄFER & GROSSMANN, 1910, SCHÄFER & REICHE, 1911) and perfect agreement between theory and experiment was found in work on the scattering of 22 cm radio waves by water cylinders (SCHÄFER & MERZKIRCH, 1923).

The mathematical formulation of these discussions is immediately useful for the present problem, but the final conclusions for metal wires or dielectric rods must not be used. It might be tempting, as a first approximation, to compare meteor trails with metal rods, since an electron gas is a good conductor and a good reflector, but it will be shown that the conclusions would be entirely fallacious. The reason is that the phase difference between electric force and current is not the same for the two media, and the kinetic energy of the electrons is important in the electron gas, but not in a metal.

Another difference is that in the ionosphere diffusion will cause a gradual transition from the ionised gas to vacuum, and the dielectric properties will vary continuously. Methods for dealing with this kind of complication are known in the theory of atomic scattering (cf. MOTT & MASSEY, 1949) which is built on the scattering of de Broglie waves in smooth potential distributions. The reflection of radio waves by meteor trails must therefore be calculated by a combination of the methods developed for atomic collisions with the theory of scattering by homogeneous rods.

The first step is to calculate the current due to the motion of charges in the ionised gas, and it is sufficient to consider the force of the electric vector on the electrons. The LORENTZ force due to the motion in the geomagnetic field can be ignored, because the gyrofrequency is of the order of 1 Mc/sec, while the observations are carried out on frequencies between 20 and 80 Mc/sec. For similar reasons collisions can be neglected, since the electron collision frequency at heights exceeding 80 kilometres is negligible compared with the wave frequencies which are used. A rough estimate which can be verified more carefully later, shows that with available radio powers the amplitudes of the electron oscillations never exceed a small fraction of a millimetre, and it is justified to consider the velocity field $\mathbf{v}$ induced in the electron gas as homogeneous over distances of this magnitude.

The equation of motion for an electron can therefore be written

$$m\frac{\partial \mathbf{v}}{\partial t} = e\,\mathbf{E} \tag{1}$$

where e and m are the electronic charge and mass, and $\mathbf{E}$ the electric vector. It should be noted that in eq. (1) it is not implied that the vector fields are homogeneous over distances comparable with a wave length, so the theory is free from any restrictions inherent in a ray approximation.

In the following discussion we assume that all vectors in the radiation field depend on the time through a factor $\exp(i\,2\,\pi\,\nu\,t)$ only, and the vector notation will be reserved for the space-dependent part of the fields.

If N is the number density of electrons, the current density due to the force in eq. (1) will be

$$\mathbf{j} = N\,e\,\mathbf{v} = (N\,e^2/i\,2\,\pi\,\nu\,m)\,\mathbf{E}. \tag{2}$$

Adding this current to the displacement current, MAXWELL's equations can be written

$$\begin{gathered}\operatorname{curl}\mathbf{H} = i\,k\,\varepsilon\,\mathbf{E}\\ \operatorname{curl}\mathbf{E} = -\,i\,k\,\mathbf{H}\\ \varepsilon = 1 - N\,e^2/\pi\,m\,\nu^2\end{gathered} \tag{3}$$

where $k = 2\,\pi/\lambda$, λ is the vacuum wave length of the electromagnetic wave, and ε is the dielectric constant of the ionised gas. ε depends on the electron density N, which is a function of the distance from the axis of the ionised column. ε is equal to unity in the vacuum sufficiently far away, but less than unity and possibly zero or negative inside the ionised column. It has been shown by DARWIN (1943), that the omission of the LORENTZ factor is correct in the expression for the dielectric constant of an ionised gas.

By this approach we include from the outset the conduction currents due to the electrons, which is equivalent to taking into account the mutual influence of the electrons on each other during the scattering. The present formulation by means of LORENTZ' theory of electrons will therefore give the answer to the question of how far BLACKETT and LOVELL's formula is valid.

For the analytical discussion we introduce a rectangular system of coordinates with z-axis along the axis of the ionised column, and assume the incident wave to come in from the positive x-axis. The electron density N depends only on the distance R form the z-axis.

The scalar equations which correspond to the system (3) form two independent groups. Three of the equations contain only the z-component E_z of the electric vector and the x- and y-components of the magnetic vector, while the other three contain only the magnetic z-component and the electric x- and y-components. The electric vector may be considered as the more important and with emphasis on the electric vector, solutions of the first group will be called *parallel* oscillations, and solutions of the second group *sagittal* oscillations, in accordance with terms commonly used in earlier work.

In the parallel oscillations the electric field lines are straight lines parallel with the ionised column, while the magnetic field lines form more complicated patterns in planes normal to the cylinder. In the sagittal oscillations the situation is reversed, the magnetic field lines being straight and parallel with the cylinder, while the electric field lines are more complicated.

For parallel scattering, therefore, we eliminate the magnetic vector in the eq. (3), which gives for the electric vector

$$\nabla^2 E_z + k^2 \varepsilon E_z = 0. \tag{4}$$

Similarly, for sagittal scattering the magnetic vector is the simpler, and we find after elimination of the electric vector

$$\nabla^2 H_z + \left(\frac{\nabla \varepsilon}{\varepsilon} \times \operatorname{curl} \mathbf{H}\right)_z + k^2 \varepsilon H_z = 0. \tag{5}$$

For a cylindrical distribution, where ε depends only on R, eq. (5) is reduced to

$$\nabla^2 H_z - \frac{1}{\varepsilon}\frac{d\varepsilon}{dR}\frac{\partial H_z}{\partial R} + k^2 \varepsilon H_z = 0. \tag{6}$$

The difference between eqs. (4) and (6) is important. From the mathematical point of view it is the counterpart to the difference in boundary conditions for electric and magnetic vectors at a sharp surface, and must appear in the differential equations when we treat a continuous distribution. It is at once apparent that the sagittal scattering must be studied with care when the dielectric constant becomes negative, because a new singularity enters into eq. (6) which is not present in eq. (4).

From a physical point of view eq. (4) shows that in parallel scattering the important effect is the current induced in the cylinder. The current appears first in the dielectric constant in the eq. (3), and is from there carried over into the term $k^2 \varepsilon E_z$ in eq. (4). On the other hand, eq. (6) shows that in sagittal scattering both currents and space charges appear. As before, the currents appear in the last term. The space charges are induced in the inhomogeneous boundary region of the plasma, and are represented by the second term which contains the derivative $d\varepsilon/dR$, which is significant in the boundaries, where the dielectric constant varies most rapidly.

Solutions for small electron densities

When the electron density is sufficiently small, approximate solutions for both types of scattering can be found, and if furthermore the cylinder diameter is small, we expect the solutions to coincide with the formula of BLACKETT and LOVELL.

We consider first the case of parallel scattering. Eq. (4) can be written

$$\nabla^2 E_z + k^2 E_z = 4\pi \varrho N E_z \tag{7}$$

where $\varrho = e^2/mc^2 = 2 \cdot 8 \times 10^{-13}$ cm.

When the electron density N is small enough, the right hand side of eq. (7) can be treated as a perturbation, and an approximate expression be substituted for E_z in that term. The equation then becomes an inhomogeneous wave equation with constant coefficients in the homogeneous part, and a solution can be found by means of GREEN's theorem. The result is found in much the same way as BORN's approximation is derived in the theory of atomic collisions (cf. MOTT and MASSEY, Ch. VI, § 4 and Ch. VII, § 1), with minor differences which are due to the cylindrical symmetry in the present case instead of spherical symmetry in atomic fields. For an ionised column, where the electron density is independent of the z-coordinate, the appropriate GREEN's function is a cylinder function of order zero, $C(k|\mathbf{R}|)$, where $|\mathbf{R}|$ is the projected distance on the xy-plane between a point inside the ionised cylinder and the distant point where the echo is observed.

With this notation the solution of eq. (7) will be

$$E_z' = \exp(ikx') - 4\pi\varrho \frac{\iiint C N E_z \, dv}{\iint \nabla C \cdot d\mathbf{S}}. \tag{8}$$

On the left hand side E_z' is the electric vector at the point of observation with coordinates $(x', 0, 0)$ outside the ionised column. On the right hand side the first term represents the incoming plane wave which is supposed to have unit amplitude, and the second term represents the scattered wave. In the second term the volume integral in the numerator is extended over a cross section of the ionised cylinder with unit thickness. The surface integral in the denominator is extended over a narrow cylinder of unit length round the point of observation. Its axis is parallel to the z-axis, and the surface element $d\mathbf{S}$ is pointing inwards towards its centre.

The second term which represents the scattered wave derives from the inhomogeneous part of eq. (7), and we may introduce a suitable approximation for E_z. For dilute ionisation and a distant point of observation on the x-axis we replace E_z by the undisturbed value

$$E_z \approx e^{ikx}$$

The correct form of the cylinder function C is determined by the condition that the scattered wave shall be a *diverging* wave at large distance from the ionised column. C must then be chosen as a Hankel funktion of the second kind and order zero, i.e.

$$C = J_0 - i\,Y_0$$

where J_0 and Y_0 are Bessel functions of the first and second kind. In the numerator of the expression for the scattered wave the argument $k\,|\mathbf{R}|$ in the Hankel function is large and approximately equal to $k\,(x' - x)$, so we can use the asymptotic expression

$$C \approx \sqrt{2\,i/\pi\,k\,x'}\ \exp\,[-\,i\,k\,(x' - x)].$$

In the denominator, where the argument is tending to zero, the main term

$$C \approx -\,(2\,i/\pi)\,\log\,|\mathbf{R}|$$

is sufficient.

Inserting these expressions for E_z and C, we find for the amplitude of the scattered wave in parallel scattering

$$E_{sc} = \exp\,(-\,i\,k\,x')\sqrt{2\,\pi/i\,k\,x'}\ \varrho \iiint e^{2ikx}\,N\,d\,v. \tag{9}$$

The sagittal scattering can be discussed along similar lines. When the electron density is sufficiently small, $\nabla\,\varepsilon$ can be used as a first order approximation for the term $\nabla\,\varepsilon/\varepsilon$ in eq. (5). Eq. (5) can then be written

$$\nabla^2 H_z + k^2 H_z = 4\,\pi\,\varrho\left[N\,H_z + \frac{1}{k^2}(\nabla N \times \text{curl } \mathbf{H})_z\right]\cdot \tag{10}$$

Comparing eqs. (7) and (10) we notice that the sagittal scattering differs from the parallel only in the last factor on the right hand side. It follows from this, that if eq. (10) is solved by GREEN's method and the approximations used for parallel scattering are introduced, the expression for the scattered wave will be the same, except for the integral

$$\iiint e^{2ikx}\,N\,d\,v$$

which will be replaced by

$$\iiint\left[e^{2ikx}\,N + \frac{1}{k^2}\,e^{ikx}\,(\nabla N \times \text{curl } e^{ikx}\,\hat{\mathbf{z}})_z\right]d\,v. \tag{11}$$

$\hat{\mathbf{z}}$ is a unit vector along the z-axis.

The integral (11) can be written

$$\iiint\left(e^{2ikx}\,N - \frac{i}{k}\,\frac{\partial N}{\partial x}\,e^{2ikx}\right)d\,v. \tag{12}$$

A partial integration with respect to x in the last term then shows that the last term is equal to twice the first term, but with opposite sign, and hence

$$E_{sc}\ (\text{parallel}) = -\,H_{sc}\ (\text{sagittal}). \tag{13}$$

The result proves that for sufficiently small electron densities, the amplitude of the echo is the same for parallel and sagittal scattering. The phase reversal shown by the minus sign in eq. (13) is only due to the use of the electric vector for the calculation of parallel scattering and the magnetic vector for the sagittal scattering. If the electric vector has the same phase and plane of polarisation in the echo as in the incident wave, the magnetic vectors must have opposite signs in order to reverse the Poynting vector from the direction of the incident wave to the opposite direction for the echo.

The response of an ionised column with small electron density is therefore entirely independent of the orientation of the plane of polarisation relative to the axis of the column. This is consistent with the theory of BLACKETT and LOVELL (1941). In their theory, the scattering of a single free electron is independent of the plane of polarisation, and the subsequent summation of contributions by electrons along a straight line does not introduce any polarisation effect. It is interesting to note, however, that the inclusion of the space charge term gave rise to the minus sign in eq. (13), so even for small electron densities the difference in the physical effects discussed before for the two types of scattering must be included in order to arrive at the right result.

It is, of course, a simple matter to derive the expression (9) for the echo amplitude in a less formal way, as done by BLACKETT and LOVELL.

We are not particularly interested in the phase of the echo amplitude, so the discussion may be simplified a little by introducing an equivalent scattering cross section for energy, which removes all reference to the distance of the ionised column from the observer or to the amplitude of the incident wave. A convenient measure is the energy scattering diameter D. This is the width which a long strip equivalent with the real column must have in order to produce the same echo, if the strip absorbs all radiation incident from the primary wave and reradiates it with axial symmetry. With unit amplitude for the primary wave, and amplitude E_{sc} in the echo at a distance x', this gives as a definition for D

$$D = 2\pi x' E_{sc}^2. \tag{14}$$

Eq. (9) then shows that the energy scattering diameter for a column with small electron density is

$$D = \frac{\lambda}{2\pi}\left[2\pi\varrho\iiint N \exp(2ikx)\,dv\right]^2. \tag{15}$$

For a cylinder so thin that the integrand is constant over the cross section, we find

$$kD = (2\pi\varrho a)^2 \tag{16}$$

where a is the number of electrons per unit length of the ionised column, and $\varrho = e^2/mc^2$.

To show that eq. (16) is identical with the result of BLACKETT and LOVELL (1941, cf. LOVELL & CLEGG, 1948) we note that the cross section of an electron for backward scattering is $4\pi\varrho^2$. The number of electrons on a straight line scattering a plane wave coherently to an observer at distance x', is $a\sqrt{x'\lambda}$, so the energy flux at the observer for unit flux in the primary wave is given by

$$4\pi \varrho^2 (a\sqrt{x'\lambda})^2/4\pi x'^2.$$

The scattering diameter is found by multiplying with the circumference $2\pi x'$, which gives

$$k D = (2\pi \varrho a)^2.$$

The result is the same as eq. (16). It shows that BLACKETT and LOVELL's theory is applicable to ionised columns which are so dilute that the incident wave can pass nearly undisturbed into the column, and so thin that phase differences over the cross section of the column are unimportant.

The slightly more general formula (15) which takes into account the finite diameter of the column, might also have been developed by BLACKETT and LOVELL's arguments without much difficulty.

For the application to meteor echoes it is of interest to study more closely the effect of the trail diameter on the echo amplitude. When a column expands from a very small diameter to its actual dimensions without change in the total number of electrons, eq. (15) shows that the amplitude of the echo is changed by a factor

$$S = \iiint N \exp(2ikx)\, dv \Big/ \iiint N\, dv. \tag{17}$$

The term $\exp(2ikx)$ in the numerator of eq. (17) represents the effect of destructive phase differences between contributions from different parts of a cross section, and the ratio S will always be less than unity under conditions when BORN's approximation is valid.

Noting that

$$\int_0^{2\pi} \exp(i\, 2kR\cos\varphi)\, d\varphi = 2\pi J_0(2kR)$$

eq. (17) can be written

$$S = \int_0^\infty N J_0(2kR) R\, dR \Big/ \int_0^\infty N R\, dR. \tag{18}$$

The electron density will be greatest near the z-axis, and fall off in some regular way with increasing distance. The value of S depends on the shape of the electron distribution N, and on the effective radius a of the column. To reach more precise conclusions S will be calculated for the following four distributions:

Case 1. Uniform electron density inside a distance $R = a$, no electrons outside. We find

$$S = \int_0^a J_0(2kR) R\, dR \Big/ \int_0^a R\, dR = J_1(2ka)/ka$$

where J_1 is a Bessel function of order one.

Case 2. Parabolic distribution $N = 1 - (R/a)^2$ for $R < a$, no electrons outside. We find

$$S = \frac{1}{(k a)^2} \int_0^{2ka} t J_0(t)\, dt - \frac{4}{(2 k a)^4} \int_0^{2ka} t^3 J_0(t)\, dt.$$

But

$$\int t J_0(t)\, dt = t J_1(t)$$

and a partial integration shows that

$$\int t^2 t J_0(t)\, dt = t^3 J_1(t) - 2 \int t^2 J_1(t)\, dt = t^3 J_1(t) - 2 t^2 J_2(t)$$

which gives

$$S = 2 J_2(2 k a)/(k a)^2.$$

Case 3. Exponential distribution $N = \exp(-R/a)$ out to infinity. Starting from the integral of LIPSCHITZ (cf. WATSON, § 13.2)

$$\int_0^\infty e^{-pt} J_0(q t)\, dt = 1/\sqrt{p^2 + q^2}$$

a differentiation with respect to p shows that

$$\int_0^\infty e^{-pt} J_0(q t)\, t\, dt = 1/(p^2 + q^2)^{3/2}.$$

Inserting this in the eq. (18) for S, we find

$$S = 1/[1 + (2 k a)^2]^{3/2}.$$

Case 4. Gaussian distribution $N = \exp(-R^2/a^2)$ out to infinity. From WEBER's first exponential integral in the theory of Bessel functions we find (cf. WATSON, § 13.3)

$$S = \exp(-k^2 a^2).$$

The four cases are illustrated in Figs. 5, 6, 7 and 8. The left hand side of each figure shows the electron distribution as function of distance from the axis, the right hand side shows the factor S with the square of the ratio $k a = 2 \pi a/\lambda$ as abscissa. The abscissa has been chosen proportional to a^2 in order to show on a *linear* time scale the variation of S for a column expanding by diffusion, when the effective radius a is increasing as the square root of time.

S is the factor with which BLACKETT and LOVELL's amplitude must be multiplied in order to correct for the finite cross section of the ionised column. It depends only on the ratio a/λ between the effective diameter and the wave length, and is significantly smaller than unity when the radius is greater than $\lambda/2\pi$.

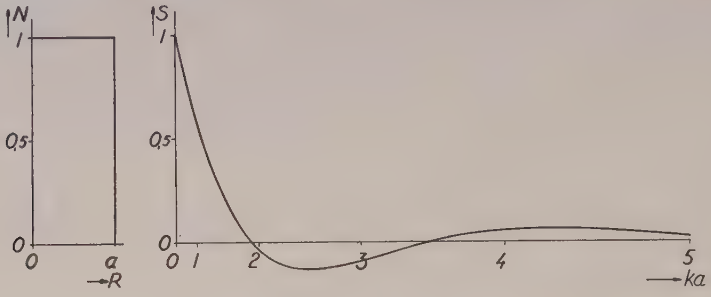

Fig. 5. Uniform distribution.

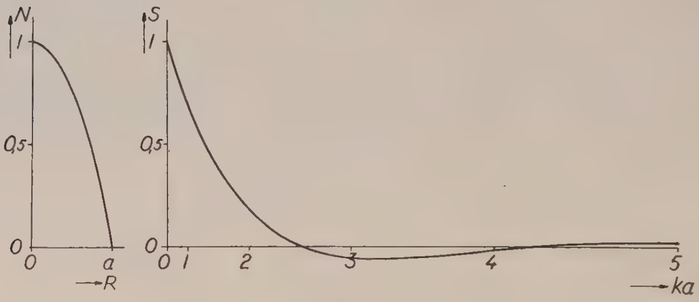

Fig. 6. Parabolic distribution.

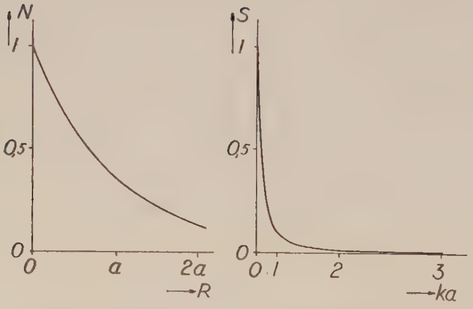

Fig. 7. Exponential distribution.

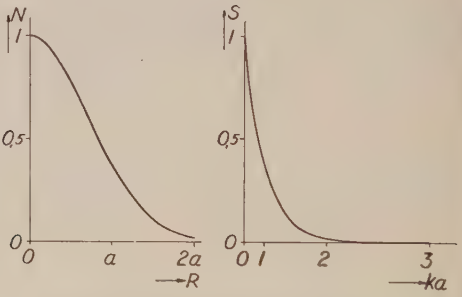

Fig. 8. Gaussian distribution.

The left hand side of each figure shows the shape of the electron distribution as function of distance from the axis, the right hand side the amplitude variations when the corresponding column expands without recombination.

There are radio observations available which illustrate both the dependence on wave length and on effective radius. Very few meteors are observed on wave lengths much shorter than 3 or 4 metres (cf. LOVELL & CLEGG, 1948). Most likely, this is due to destructive interference of the contributions to the echo from various parts of the cross section. For a wave length of, say, 1.5 metres on which observations have been done, the significant distance

$\lambda/2\pi$ is about 25 centimetres. Considering the mean free paths in the upper atmosphere, we then expect $2\pi a/\lambda$ to be of the order of unity, or greater, in many cases and the dielectric constant to differ only slightly from unity. The importance of the factor S is then evident, particularly for a Gaussian distribution.

Since a meteor trail expands by diffusion, we can also trace the effect of an increasing radius. In the discussion of the ionised trails produced by meteors it was pointed out that expansion will soon reduce the echo amplitude, and predictions about echo duration based on this view (HERLOFSON, 1948) have been verified in some cases (LOVELL, 1948).

It is of interest to decide whether the echo from an expanding column will decay regularly, or fluctuate with diminishing amplitude. Figs. 5 and 6 show that fluctuations will occur if the boundary of the column is so sharp that a definite front side and back side can be located. On the other hand, the electron distributions in Figs. 7 and 8 are associated with a regular decay, because the boundaries are so diffuse that the column has no well defined surface. This gradual transition from a fluctuating decay to a regular decay is a property of the sharpness of the surface only, and can be shown even more readily for reflections by a semitransparent plane layer than for the cylindrical case, as the Bessel functions would then be replaced by trigonometrical functions.

Diffusion will nearly at once transform any narrow electron distribution into a Gaussian distribution, so when an actual meteor trail has become semitransparent, the echo will decay regularly as shown in Fig. 8. It has been observed (MILLMAN & MCKINLEY 1949, ELLYETT 1950, GREENHOW 1950) that echoes from meteor trails fluctuate strongly, and there has been a certain amount of discussion whether this is due to trail distortions by turbulence in the atmosphere or due to interference between reflections from an effective front and back side of an expanding column. The conclusion of the present discussion supports the view that the flutter of meteor echoes is not due to interference inside the meteor trail, but is caused by distortions of the straight trail in variable winds.

Parallel scattering for large electron densities

In parallel scattering the electric vector will induce currents which run along the surfaces of constant electron density. No space charge will be induced, and the currents only are responsible for the distortion of the electromagnetic field inside the ionised column. This is the same physical situation as occurs when a plane wave is incident normally on a plane stratification of ionisation, and with very little effort we can solve the problem of parallel scattering by means of this comparison.

We consider first an unlimited homogeneous region with a negative dielectric constant ε. The electric vector will satisfy the wave equation

$$\nabla^2 E + k^2 \varepsilon E = 0$$

and a plane solution with attenuation in the direction of the x-coordinate is

$$E = \exp(-k\sqrt{-\varepsilon}\, x). \tag{19}$$

The amplitude decreases by a factor $1/e = 37\ \%$ over a distance

$$d = 1/k\sqrt{-\varepsilon} = 1/k\sqrt{-1 + \frac{N e^2}{\pi m \nu^2}} > \frac{\nu}{e k}\left(\frac{\pi m}{N}\right)^{1/2}. \tag{20}$$

The distance d is similar to the depth of penetration in the theory of skin effect in metals. If the homogeneous region has a plane boundary it shows how far inside the boundary an incident plane wave would go before the amplitude is appreciably modified.

Since we have ascertained that the physical effects are the same in parallel scattering by ionised cylinders as for normal reflection from a plane ionised layer, we expect the wave to penetrate into the cylinder at least as deep as the distance d without much attenuation.

The essence of the BORN approximation used for the case of small electron density is the assumption that the primary wave is not appreciably modified inside the cylinder, and the approximation may be used for the present case of large electron densities if the cylinder is so thin that the radius a is smaller than the depth of penetration d. According to eq. (20), this condition is satisfied if

$$a < \frac{\nu}{e k}\left(\frac{\pi m}{N}\right)^{1/2}$$

or

$$\alpha = N \pi a^2 < 1/4\varrho \tag{21}$$

where α is the number of electrons per cm length of the cylinder, and $\varrho = e^2/m c^2$.

Inserting numerical values, eq. (21) shows that BORN's approximation, can be used in parallel scattering both for small and large electron densities if the number of electrons per cm length is smaller than 10^{12}. The theory of meteor ionisation indicates that this condition is satisfied for most of the ordinary shooting stars. Since a wave penetrates more easily across a diffuse boundary than a sharp one, BORN's approximation can be expected to be valid for meteor trails even somewhat beyond the limit in eq. (21).

From the point of view of BLACKETT and LOVELL's theory it is interesting that in the case of *parallel* scattering there is, for a useful range of the parameters, no conflict between the two assumptions of the theory, namely that the diameter of the column shall be small and the electrons not disturb each other appreciably.

The conclusion of this discussion has been checked by the exact methods developed for sagittal scattering in the next section. It is confirmed that with less than 10^{12} electrons per cm length of the cylinder, the approximation is valid to within a few per cent.

Sagittal scattering for large electron densities

The theory of sagittal scattering by a dense column is more complicated than the calculation of parallel scattering.

To arrive at a simple solution one might try a comparison with a corresponding plane problem. It is characteristic for sagittal scattering that the electric vector has a component across the surfaces of equal electron density, which gives rise to a space charge in the boundaries of the column, in addition to the induced current. In a plane problem a similar situation arises when a plane wave is reflected at an oblique angle, with a polarisation which directs the electric vector across the planes of equal electron density. Unfortunately, in the wave treatment of the plane problem this case has been avoided because it leads into difficulties (HARTREE, 1929), and in the ray approximation space charge in a transverse wave is excluded from the outset.

For these reasons it seems inevitable that the more formal theory of scattering by cylindrical obstacles must be applied to the sagittal case. The general approach is very similar to the wave mechanical theory of the scattering of particles by a centre of force (cf. MOTT & MASSEY, 1949, Ch. II), except for the use of cylinder coordinates in the present case and spherical coordinates in wave mechanics.

As before, the sagittal scattering is discussed by means of the magnetic vector H_z, which satisfies eq. (6)

$$\nabla^2 H_z - \frac{1}{\varepsilon}\frac{d\varepsilon}{dR}\frac{\partial H_z}{\partial R} + k^2 \varepsilon H_z = 0. \tag{6}$$

We introduce cylinder coordinates R, φ where φ is the angle round the z-axis, reckoned from the x-axis. Eq. (6) gives then

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial H_z}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 H_z}{\partial \varphi^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dR}\frac{\partial H_z}{\partial R} + k^2 \varepsilon H_z = 0. \tag{22}$$

H_z can be expanded in a Fourier series of the angle φ, with coefficients f_n which depend on R. As the primary wave is incident along the direction of the x-axis, the wave field will be symmetrical about the xz-plane, and it is sufficient to retain the cosine terms in the Fourier series. Hence

$$H_z = \sum_{n=0}^{\infty} f_n(R)\cos n\varphi. \tag{23}$$

Introducing the series (23) into eq. (22), we obtain

$$f_n'' + \left(\frac{1}{R} - \frac{\varepsilon'}{\varepsilon}\right) f_n' + \left(k^2 \varepsilon - \frac{n^2}{R^2}\right) f_n = 0 \tag{24}$$

where $n = 0, 1, 2 \ldots$.

When $R \to \infty$, $\varepsilon \to 1$ and eq. (24) becomes asymptotically

$$f_n'' + f_n'/R + (k^2 - n^2/R^2) f_n = 0. \tag{25}$$

The solution of eq. (25) is

$$f_n = a_n J_n(kR) + b_n Y_n(kR) \tag{26}$$

and this is also a solution of eq. (24) valid for large R. J_n and Y_n are Bessel functions of order n of the first and second kind. a_n and b_n are constants which may be complex.

At large distance from the z-axis the series in eq. (23) shall describe a plane incident wave and a diverging cylinder wave. The incident wave is given by

$$H_{\text{inc}} = \exp(ikx) = \exp(ikR\cos\varphi). \tag{27}$$

This expression must also be expanded in a Fourier series of the angle φ, and we find by JACOBI's series (cf. WATSON, § 2.22)

$$H_{\text{inc}} = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(kR)\cos n\varphi \tag{28}$$

where ε_n is NEUMANN's factor which is equal to 1 when $n = 0$, but equal to 2 otherwise.

The amplitude of the scattered cylinder wave can be written in the form

$$H_{\text{sc}} = h(\varphi)\exp(-ikR)/\sqrt{R} \tag{29}$$

where $h(\varphi)$ depends on the angle φ only, and determines the amplitude of the radiation scattered into the various directions. According to eq. (14), the energy scattering diameter D for backward scattering is related to $h(\varphi)$ by the formula

$$D = 2\pi\,|h(0)|^2. \tag{30}$$

At large distance from the z-axis the sum of the incident wave and the scattered wave shall be identical with the asymptotic wave field determined by the solutions eq. (26), i.e.

$$H_{\text{inc}} + H_{\text{sc}} = \sum_{n=0}^{\infty} [a_n J_n(kR) + b_n Y_n(kR)]\cos n\varphi. \tag{31}$$

Eq. (31) is only valid for large R, and we can introduce the asymptotic expansions for the Bessel functions

$$\begin{aligned} J_n(kR) &\approx \sqrt{2/\pi kR}\cos[kR - \pi(2n+1)/4] \\ Y_n(kR) &\approx \sqrt{2/\pi kR}\sin[kR - \pi(2n+1)/4]. \end{aligned} \tag{32}$$

We insert the asymptotic expansions (32) and the expressions (28) and (29) into eq. (31). In the resulting equation we replace the trigonometrical functions by complex exponential functions, and compare separately the coefficients for $\exp(ikR)$ and $\exp(-ikR)$ in the left hand and right hand sides. This gives the following relations

$$a_n - i b_n - \varepsilon_n i^n = 0 \qquad n = 0, 1, 2 \ldots \tag{33}$$

$$\exp(-i\pi/4)\sqrt{2\pi k}\,h(\varphi) = \sum_{n=0}^{\infty} (a_n + i b_n - \varepsilon_n i^n)\, i^n \cos n\varphi. \tag{34}$$

Eqs. (33) are one set of relations between the constants a_n and b_n, and show that the boundary condition for large R determines one of the two arbitrary constants in each of the solutions (26). Only one constant of integration is therefore left free in eq. (24). This constant is determined by the requirement that f_n shall be finite when $R \to 0$.

It will be seen presently that this second condition gives uniquely the ratio f'_n/f_n as function of R. Taking this for granted, a comparison with eq. (26) shows that for large R

$$\frac{f'_n}{f_n} = k\frac{(a_n/b_n)\,J'_n(kR) + Y'_n(kR)}{(a_n/b_n)\,J_n(kR) + Y_n(kR)} \tag{35}$$

which means that the ratio

$$c_n = a_n/b_n \tag{36}$$

is uniquely determined if f'_n/f_n is known for large R. Accepting c_n as known, a_n and b_n can then be found from the eqs. (33) and (36). Inserting the result in eq. (34) we obtain for $h(\varphi)$

$$\exp(-i\,3\pi/4)\sqrt{\pi k/2}\;h(\varphi) = \sum_{n=0}^{\infty}\frac{\varepsilon_n(-1)^n}{c_n - i}\cos n\varphi. \tag{37}$$

According to eq. (30) this gives for the energy scattering diameter for backward scattering

$$D\frac{2\pi}{\lambda} = 4\left|\sum_{n=0}^{\infty}\frac{\varepsilon_n(-1)^n}{c_n - i}\right|^2 \tag{38}$$

where $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n = 1, 2, 3 \ldots$.

Eq. (38) shows that the scattering diameter D is fully determined by the coefficients c_n. There is a close connection between c_n and the phase angles commonly used in the theory of atomic scattering, but for the present discussion c_n will be more convenient than phase angles.

It now remains to show that c_n is determined by the requirement of convergence for f_n when $R \to 0$. The discussion proceeds along the same lines as the calculation of phase angles in the theory of atomic scattering. The eq. (24) has a regular singularity at the origin with one regular solution which starts with a term R^n, and one diverging solution. The appropriate solution for f_n is therefore determined, apart from a constant factor. This means that f'_n/f_n is uniquely determined and can be calculated for all values of R by integration of eq. (24), which is sufficient to determine c_n from eq. (35).

There is, however, one difficulty which arises when ε is negative for small R, because we must then pass a zeropoint $R = R_0$ for ε on the way out from the origin. This gives rise to a second singularity in eq. (24). Before the integration can be continued outwards to large distances where $\varepsilon = 1$, it must be discussed how the integration up to a point right inside the singularity can be continued outside the singularity. For this purpose we shall regard R as a complex variable, and study the region in the complex plane round the point $R = R_0$ on the real axis. This is a natural mathematical approach, and it would immediately have suggested itself if collision damping had been included and the dielectric constant had been complex.

In the region round the zeropoint $R = R_0$, ε can be expanded in a power series in $s = R - R_0$,

$$\varepsilon = \varepsilon' s + \tfrac{1}{2} \varepsilon'' s^2 + \cdots. \tag{39}$$

We find then

$$-\frac{\varepsilon'}{\varepsilon} = -\frac{1}{s} + \text{const.} + O(s). \tag{40}$$

For small values of s eq. (24) assumes the form

$$f_n'' + (-1/s + A_0 + A_1 s + \cdots) f_n' + (-n^2/R_0^2 + B_1 s + B_2 s^2 + \cdots) f_n = 0 \tag{41}$$

where $A_0, A_1 \ldots$ and $B_1, B_2 \ldots$ are constants.

Following FROBENIUS' method, we find the indicial equation

$$r(r-1) - r = 0 \tag{42}$$

which has the roots $r = 0$ and $r = 2$. One of the solutions in the region round $s = 0$ will be a power series u_n starting with the higher of the two exponents

$$u_n = s^2 + C_3 s^3 + C_4 s^4 + \cdots \tag{43}$$

where $C_3, C_4 \ldots$ are constants, determined by the coefficients in eq. (41).

The difference between the roots of the indicial equation (42) is an integer, and according to the general theory for solutions near a regular singularity, another solution v_n will have the form

$$v_n = u_n \log s + D_0 + D_1 s + D_2 s^2 + \cdots \tag{44}$$

where $D_0, D_1, D_2 \ldots$ are constants which can be determined by substitution in eq. (41). To the last part of the solution v_n can always be added an arbitrary multiple of u_n, so the coefficients from D_2 onwards will all contain an undetermined additive term.

For the general discussion D_0 and D_1 are wanted. Substituting eq. (44) into eq. (41), and noting that u_n is also a solution of eq. (41), we find when grouping together terms with the same power of s

$$-D_1/s + (2 + D_1 A_0 - D_0 n^2/R_0^2) + O(s) = 0. \tag{45}$$

Eq. (45) gives

$$D_1 = 0 \quad \text{and} \quad D_0 = 2 R_0^2/n^2. \tag{46}$$

This shows that the general solution of eq. (41) near the zeropoint for ε at $R = R_0$ can be written

$$f_n = (s^2 + C_3 s^3 + C_4 s^4 + \cdots)\left(\frac{n^2}{2 R_0^2} P \log s + Q\right) + (1 + D_3 s^3 + D_4 s^4 + \cdots) P \tag{47}$$

where P and Q are two arbitrary constants of integration, and the other coefficients are determined by the differential equation.

For $n = 0$ the logarithmic term in eq. (47) vanishes. The singularity is only accidental, and appears because the constants of integration are determined by the values of the function and its second derivative at $R = R_0$, while the first derivative is identically zero.

When n differs from 0, the logarithmic term appears in the general solution. This is harmless for the magnetic field which will be finite, because f_n is bounded, but the situation is more difficult for the electric field and the velocity. The tangential component of the electric vector is proportional to $(1/\varepsilon)\,\partial H_z/\partial R$, which gives rise to terms containing f_n'/s in the corresponding Fourier series, and eq. (47) shows that the largest term in f_n'/s for small s will be proportional to $\log s$. We are, however, compelled to accept this divergence. Of the two constants of integration available for the eq. (24), one is determined by the amplitude at large distance as shown in the discussion of eqs. (33), and the other is determined by the behaviour of f_n near the origin. We are therefore not in a position to choose a third constant of integration, and put $P = 0$ in eq. (47).

The dilemma can be settled by a comparison of the order of the divergence for $R = 0$ and $R = R_0$. For $R = 0$ the inadmissible solution diverges as R^{-n}, when $n > 0$, whereas the divergence for $R = R_0$ is only logarithmical. This means that the general solution near the origin is not even quadratically integrable, so the electrical and mechanical energy would be infinite, while the general solution near $R = R_0$ is quadratically integrable, and will in actual cases only give rise to a negligible amount of electrical or mechanical energy. The general solution can therefore be used in the region near $R = R_0$.

Less formally, it is also worth noting that the value of the logarithmic term is appreciable only in a layer so thin that it is hardly significant for the physical system in any practical case.

It is obviously important that $D_1 = 0$, as shown in eq. (46), otherwise f_n' would not vanish, and the electric field would not have been quadratically integrable.

We are now in a position to connect the value of f_n'/f_n inside $R = R_0$ with a similar value outside, and determine the coefficients c_n in the scattering formula. This might be done directly by means of eq. (47), but it may be convenient to follow a slightly different method. For the actual calculation of c_n, particularly by numerical methods, it is advantageous to use a Riccati equation which corresponds to eq. (24). Introducing

$$y_n = f_n'/\varepsilon f_n \tag{48}$$

substitution in eq. (24) gives

$$y_n' + \varepsilon y_n^2 + y_n/R + k^2 - n^2/R^2\varepsilon = 0. \tag{49}$$

Knowing the behaviour of f_n and hence of y_n near $R = R_0$, we can now integrate eq. (49) from a point A where the dielectric constant is negative, to a point B where it is positive, along the path in the complex plane shown in Fig. 9. When A and B converge towards R_0, the only non-vanishing con-

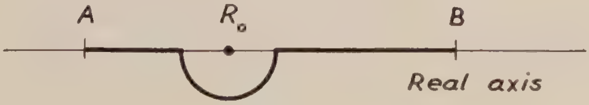

Fig. 9. Path of integration in the complex R-plane bypassing the singularity $R = R_0$.

tribution to the integration comes from the last term in eq. (49) and we obtain

$$y_n(B) - y_n(A) = \frac{n^2}{\varepsilon' R_0^2}\left(i\pi + \log\left|\frac{\varepsilon_B}{\varepsilon_A}\right|\right). \tag{50}$$

$y_n(B)$ and $y_n(A)$ are the values of y_n at B and A, ε_B and ε_A are the corresponding values of ε.

The choice of a path of integration which bypasses the singularity *below* the real axis can be justified by the following physical argument. If collisions between electrons and heavy particles had been taken into account and averaged over the different free paths, the equivalent equation of motion for an electron would have been (APPLETON and CHAPMAN, 1932)

$$m\dot{\mathbf{v}} + m\nu_c\mathbf{v} = e\mathbf{E}$$

where ν_c is the collision frequency. The dielectric constant would have been complex and equal to

$$\varepsilon = 1 - Ne^2 \Big/ \left(1 - \frac{i}{2\pi}\frac{\nu_c}{\nu}\right)\pi m \nu^2. \tag{51}$$

At the distance where ε should have vanished if ν_c had been zero, the derivative N' of N is negative, because the electron density is falling off with increasing R.

Writing for small values of $R - R_0$

$$N = N(R_0) + N'(R - R_0) \quad \text{where} \quad 1 - N(R_0)e^2/\pi m\nu^2 = 0$$

eq. (51) gives

$$\varepsilon = 1 - \frac{N(R_0)e^2}{\pi m \nu^2}\,\frac{1 + N'(R - R_0)/N(R_0)}{1 - i\nu_c/2\pi\nu}. \tag{52}$$

Recalling that $N' < 0$, eq. (52) shows that with a small collision frequency we should have to go a slight distance above the real axis in order to find the zeropoint for ε, and the path of integration for the wave function f_n would pass below the singularity in eq. (24). For this reason the path of integration in Fig. 9 is drawn below the singularity $R = R_0$ on the real axis in the case of negligible collision camping.

Eq. (50) shows that when $R = R_0$ is being passed, y_n goes over from real values to complex values. The coefficients c_n in the formula (38) for the scattering diameter will therefore be complex, and an equivalent formula for D by means of phase angles would have lead to complex angles.[1] This is one reason why phase angles are less useful in the present problem. It will also be seen that resonances occur, which makes interpolation in phase angles less convenient.

For a cylinder with a sharp transition between the regions of negative and positive dielectric constant, $\varepsilon' \to \infty$, and y_n is continuous across the boundary.

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[1] The complex angles are a consequence of the linear approximation in eq. (1), and the nonlinear terms would probably be required to obtain agreement with unpublished work by BOHR, PEIERLS and PLACZEK (cf. MOTT and MASSEY, 1949, p. 133).

In this particular case the boundary condition is the same as for any dielectric, but otherwise, when the zeropoint for ε is located in a diffuse boundary region, y_n is discontinuous.

Eq. (24) or the equivalent Riccati equation (49) can only be solved by numerical methods, except in the case of a homogeneous cylinder. If ε is constant inside a cylinder, the solution of eq. (24) between the axis and the surface is

$$f_n = J_n(\sqrt{\varepsilon}\, k R). \tag{53}$$

Outside the cylinder, the vacuum solution eq. (26) is valid, and

$$f_n = a_n J_n(kR) + b_n Y_n(kR). \tag{54}$$

Since the boundary is sharp, y_n is continuous across the surface of the cylinder. Calling the cylinder radius a, we find

$$y_n = \frac{k\sqrt{\varepsilon}}{\varepsilon} \frac{J_n'(\sqrt{\varepsilon}\, k a)}{J_n(\sqrt{\varepsilon}\, k a)} = k \frac{c_n J_n'(ka) + Y_n'(ka)}{c_n J_n(ka) + Y_n(ka)} \tag{55}$$

and it is possible to calculate c_n for all values of n.

For a first survey, we are particularly interested in cylinders where $ka = 2\pi a/\lambda$ must be considerably smaller than unity to make the dielectric constant negative. Since

$$\varepsilon = 1 - N e^2/\pi m \nu^2 = 1 - 4 a \varrho/(ka)^2 \tag{56}$$

where $\varrho = e^2/mc^2$, this restriction is equivalent to

$$4 a \varrho \ll 1 \quad \text{or} \quad a \ll 10^{12} \text{ electrons/cm} \tag{57}$$

which is obeyed in most meteor trails.

The calculation of c_n is slightly different for the cases $n = 0$ and $n > 0$.

For $n = 0$, eq. (55) gives

$$\frac{1}{\sqrt{\varepsilon}} \frac{J_1(\sqrt{\varepsilon}\, k a)}{J_0(\sqrt{\varepsilon}\, k a)} = \frac{c_0 J_1 + Y_1}{c_0 J_0 + Y_0} \tag{58}$$

where the omitted arguments in the Bessel functions are ka. The argument $\sqrt{\varepsilon}\, k a$ in the left hand side of eq. (58) can be written

$$\sqrt{\varepsilon}\, k a = \sqrt{(ka)^2 - 4 a \varrho}$$

which shows that it is always a small purely imaginary number when $\varepsilon < 0$ and the condition (57) is fulfilled. Calling the left hand side of eq. (58) S_0, we replace the Bessel functions by the first terms in their power series, and obtain approximately

$$S_0 \approx \frac{1}{\sqrt{\varepsilon}} \frac{1}{2} \sqrt{\varepsilon}\, k a = \frac{1}{2} k a. \tag{59}$$

Solving eq. (58) for c_0, we find

$$c_0 = \frac{Y_1 - Y_0 S_0}{J_0 S_0 - J_1}. \tag{60}$$

For small values of ka, the first term in the numerator in eq. (60) diverges as $(ka)^{-1}$, while the second term vanishes as $(ka)\log(ka)$. The numerator is therefore large of the order $(ka)^{-1}$. In the denominator, both terms vanish as $(ka)^1$. Moreover, inserting in the denominator the expression (59) and the first approximation to the Bessel functions, we find that the first order terms cancel, so the denominator will vanish as $(ka)^3$. This shows that for small values of ka, c_0 diverges as $(ka)^{-4}$. Introducing this in eq. (38) for the energy scattering diameter, it is found that the corresponding term in the series will vanish as $(ka)^4$, and as will be seen presently, this is negligible compared with the contributions from the subsequent terms.

From a physical point of view this is quite natural because the part of the electromagnetic field which is associated with the coefficient c_0, consists of a magnetic field with axial symmetry, and an electric field with field lines running in circles round the z-axis. It is evident that the coupling is weak between this partial field and an incident wave whose fields are nearly homogeneous over the cross section of the narrow column.

Next we consider the case $n > 0$.

When n differs from 0, the derivatives of the Bessel functions in eq. (55) can be eliminated by means of the two relations

$$J'_n(z) = -(n/z)\, J_n(z) + J_{n-1}(z)$$

and

$$Y'_n(z) = -(n/z)\, Y_n(z) + Y_{n-1}(z)$$

which gives

$$\frac{1}{\sqrt{\varepsilon}} \frac{J_{n-1}(\sqrt{\varepsilon}\, ka)}{J_n(\sqrt{\varepsilon}\, ka)} + \frac{n}{ka}\left(1 - \frac{1}{\varepsilon}\right) = \frac{c_n J_{n-1} + Y_{n-1}}{c_n J_n + Y_n}. \tag{61}$$

The argument ka is omitted in all Bessel functions.

Calling the left hand side of eq. (61) S_n, we obtain for c_n

$$c_n = \frac{Y_{n-1} - Y_n S_n}{J_n S_n - J_{n-1}}. \tag{62}$$

The same reasoning as used for c_0 shows that $\sqrt{\varepsilon}\, ka$ is always small in the case under consideration, and we can write approximately

$$S_n \approx \frac{1}{\sqrt{\varepsilon}} \frac{2n}{\sqrt{\varepsilon}\, ka} + \frac{n}{ka}\left(1 - \frac{1}{\varepsilon}\right) = \frac{n}{ka}\left(1 + \frac{1}{\varepsilon}\right). \tag{63}$$

The terms ignored in eq. (63) are of the order $(ka)^1$.

For small values of ka, the dominating term in the numerator of eq. (62) is $Y_n S_n$, which is in general greater than Y_{n-1} by a factor of the order of $(ka)^2$. An exception is the case $n = 1$, when Y_0 contains a logarithmic term, but even then $Y_1 S_1$ dominates. In the dominator, the two terms are of the same order in ka. Introducing the first terms in the series for the Bessel functions, we find

$$J_n S_n - J_{n-1} \approx \left(\frac{k a}{2}\right)^{n-1} \frac{1}{2\,(n-1)\,!} \left(\frac{1}{\varepsilon} - 1\right).$$

Furthermore

$$Y_n S_n \approx -\frac{(n-1)\,!}{\pi} \left(\frac{2}{k a}\right)^{n+1} \frac{n}{2} \left(1 + \frac{1}{\varepsilon}\right).$$

An approximate expression for c_n, when ka is small and $\varepsilon < 0$, is therefore

$$c_n \approx \left(\frac{2}{k a}\right)^{2n} \frac{n\,!\,(n-1)\,!}{\pi} \frac{1+\varepsilon}{1-\varepsilon} \qquad n = 1, 2, \ldots \tag{64}$$

We consider first the limiting case of a cylinder with a given number of electrons per cm length, which is compressed to a very small diameter. ka will then vanish, ε will become infinitely large with negative sign, and the coefficients c_n will diverge as $(ka)^{-2n}$ when $n > 0$, and as $(ka)^{-4}$ when $n = 0$. Eq. (38) then shows that the energy scattering diameter will vanish as $(ka)^4$.

This is distinctly different from the behaviour of the scattering diameter for *parallel* scattering with large electron densities. The discussion of parallel scattering showed that BORN's approximation and eq. (16) are valid right down to vanishing diameters if $4 a \varrho \ll 1$, and the scattering diameter will not vanish with the diameter.

Hence, when the electron density is so great the dielectric constant has a *large* negative value, the echo from a column will be much weaker in the sagittal case than in the parallel case. From a physical point of view this is reasonable. In parallel scattering the electrons slide up and down along the cylinder, and the motion is not hindered by a great electron density. In sagittal scattering, however, the electrons move across the cylinder, and space charges which tend to restore the equilibrium, are set up along the boundaries. When the electron density becomes sufficiently great, the electrostatic forces will prevent even the slightest displacement of the charges, and the cylinder will not respond to the incident wave.

If the cylinder expands from a vanishing diameter, the dielectric constant will vary from $-\infty$ towards 0, which is the limit for the approximations of the present section. According to eq. (64) c_n will then assume real values ranging from $-\infty$ when $ka \to 0$, through $c_n = 0$ when $\varepsilon = -1$, up to positive values when ε approaches 0. This regular progression of c_n along the real axis is significant and gives rise to effects which are typical only for an ionised gas, where the dielectric constant can be real and negative.

The modulus of each term in the series in of eq. (38) is inversely proportional to the distance in the complex plane between c_n and the point i. Since c_n will pass once through the origin, this distance will go through a minimum, and a resonance peak will occur in the scattering diameter. At the same time the argument of $c_n - i$, i.e. the phase of the echo, changes nearly 180°.

In the approximation used in eq. (64) all the terms in the series for D will have their resonance peaks at $\varepsilon = -1$. If the approximation in the expression

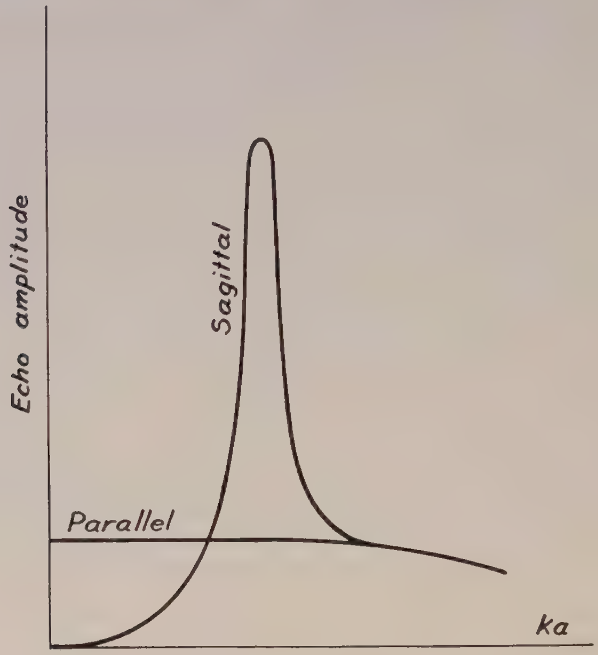

Fig. 10. Echo amplitude in both polarisations for a column which expands without recombination. Linear coordinate axes.

for c_n had been carried one step further, a very small separation between the resonance peaks would have been found, the peak for $n = 1$ falling a little below $\varepsilon = -1$, and the peaks of higher orders following in a monotonous sequence crowding towards $\varepsilon = -1$ for large orders.

At each peak the energy scattering diameter goes up to a value of the order

$$D\ (\text{resonance}) \approx (8/\pi)\,\lambda. \tag{65}$$

This value should be compared with the scattering diameter for parallel scattering which is according to eq. (16) equal to, or smaller than

$$D = (2/\pi)\,\lambda\,(\pi\,\varrho\,\alpha)^2.$$

As we are concerned with cases where $\alpha < 10^{12}$ electrons/cm length, i.e. $\pi\,\varrho\,\alpha < 1$, it is evident that the resonance peak in sagittal scattering gives rise to a stronger response than in parallel scattering, where no resonance occurs. If $\pi\,\varrho\,\alpha \ll 1$, the response in the parallel polarisation can be negligible compared with the response at resonance in sagittal scattering.

The situation is illustrated in Fig. 10, which shows for both polarisations the amplitude of the echo as function of $k a = 2\pi a/\lambda$. The amplitude is proportional to the square root of the scattering diameter. In sagittal scattering only the resonance peak corresponding to $n = 1$ has been included for the sake of clarity. The figure shows that when the column is narrow, BORN's approximation gives a good value for the amplitude in parallel scattering until phase differences over the cross section set in, while the amplitude in sagittal scattering starts from 0 for small dimensions of the column, passes through a high resonance peak, and finally approaches the values for parallel scattering when the column has expanded sufficiently.

We define the half width of one of the resonance peaks as the difference $\Delta(ka)$ between the values of ka for which the scattering diameter is one half of the maximum value. If we consider each peak separately, eq. (38) shows that D is reduced to half its maximum value when c_n is located either at -1 or at $+1$ on the real axis, because the distance to the point i is then $\sqrt{2}$ as compared with 1 when c_n is at the origin. For $\varepsilon = -1$, eq. (64) gives

$$\frac{d\,c_n}{d\,(ka)} = \left(\frac{2}{ka}\right)^{2n} \frac{n!\,(n-1)!}{\pi}\,\frac{1}{2}\,\frac{d\,\varepsilon}{d\,(ka)}. \tag{66}$$

But when $\varepsilon = -1$, $(ka)^2 = 2\,\alpha\varrho$ and hence by eq. (56)

$$\frac{d\,\varepsilon}{d\,(ka)} = \frac{4\,\alpha\varrho\,2}{(ka)^3} = \frac{4}{ka}.$$

Inserting this in eq. (66) we find

$$\frac{d\,c_n}{d\,(ka)} = \left(\frac{2}{ka}\right)^{2n+1} \frac{n!\,(n-1)!}{\pi} \tag{67}$$

valid for $\varepsilon = -1$. Using eq. (67) over the range in ka which corresponds to $-1 < c_n < 1$, we find for the halfwidth $\Delta(ka)$ of the resonance peak

$$\frac{\Delta(ka)}{ka} = \left(\frac{ka}{2}\right)^{2n} \frac{\pi}{n!\,(n-1)!} = \left(\frac{\alpha\varrho}{2}\right)^{n} \frac{\pi}{n!\,(n-1)!}. \tag{68}$$

The resonance peak can be compared with a filter with a frequency half width $\Delta\nu$ where $\Delta\nu$ can be determined by eq. (68). As $ka = 2\pi a/\lambda$,

$$\Delta\nu/\nu = \Delta(ka)/ka.$$

Eq. (68) shows then that when $\alpha\varrho \ll 1$, the width $\Delta\nu$ of the filter is a very small fraction of the frequency ν employed, and the width depends on n in such a way that it is reduced to exceedingly small values for $n > 1$. The inverse of the ratio in eq. (68)

$$\frac{\nu}{\Delta\nu} = \left(\frac{2}{\alpha\varrho}\right)^{n} \frac{n!\,(n-1)!}{\pi}$$

gives an estimate of the number of oscillations taken up by the transients after the sudden onset of a wave train with a steep front. Considering as an illustration $\alpha = 10^{10}$ electrons/cm length, the transients in the first peak with $n = 1$ would occupy about 300 oscillations. On a wave frequency of 100 Mc/sec this corresponds to 3 microseconds which elapse before the steady state solution is becoming significant. It is therefore evident that in meteor observations with pulses of some microseconds duration, the first resonance oscillation can be fairly well established during a pulse, but there is no possibility of observing the peaks of higher orders.

This is one reason for taking into account the first resonance peak only in Fig. 10 and in the subsequent discussion. Another reason, effective even for stationary oscillations, is a strong reduction in amplitude when the boundaries of the cylinder are diffuse instead of perfectly sharp. The narrow peaks of higher orders are particularly sensitive to this effect, and will virtually disappear as soon as the boundary becomes slightly diffuse.

Accurate results for inhomogeneous cylinders can only be obtained by numerical integration of eq. (24) or the equivalent Riccati equation (49), but the effect of a slightly diffuse boundary can be estimated by approximate methods. For a cylinder which is fairly homogeneous except for a thin, diffuse boundary layer, we can use the solution for homogeneous cylinders up to the inner side of the boundary layer, and then bridge the boundary layer by means of eq. (50). If the thickness of the boundary layer is small, and we are mainly interested in the case of a dielectric constant near -1 in the core of the cylinder, eq. (55) is modified by an imaginary term from eq. (50), and becomes

$$\frac{k\sqrt{\varepsilon}}{\varepsilon}\,\frac{J_n'(\sqrt{\varepsilon}\,k\,a)}{J_n(\sqrt{\varepsilon}\,k\,a)} + \frac{n^2\,i\,\pi}{\varepsilon'\,R_0^2} = k\,\frac{c_n\,J_n' + Y_n'}{c_n\,J_n + Y_n}. \tag{69}$$

In eq. (69) the radius a is now roughly the distance from the z-axis to the zeropoint for ε, and is therefore equal to R_0. We have ignored a term $\log|\varepsilon_B/\varepsilon_A|$ in eq. (69) as compared with $i\pi$, because we are mainly concerned with the case $|\varepsilon_B/\varepsilon_A| \approx 1$, when the logarithm can be ignored in an approximate survey.

Eq. (69) can be rearranged on the form of eq. (61) and gives

$$\frac{1}{\sqrt{\varepsilon}}\,\frac{J_{n-1}(\sqrt{\varepsilon}\,k\,a)}{J_n(\sqrt{\varepsilon}\,k\,a)} + \frac{n}{k\,a}\left(1 - \frac{1}{\varepsilon}\right) + \frac{n^2\,i\,\pi}{k\,\varepsilon'\,a^2} = \frac{c_n\,J_{n-1} + Y_{n-1}}{c_n\,J_n + Y_n}. \tag{70}$$

Comparing with the expression S_n in eq. (63), we see from eq. (70) that S_n has been altered by the addition of an imaginary term, and becomes

$$S_n \approx \frac{n}{k\,a}\left(1 + \frac{1}{\varepsilon}\right) + \frac{n^2\,\pi}{k\,\varepsilon'\,a^2}\,i. \tag{71}$$

S_0 is not changed at all, because the imaginary term in eq. (69) then disappears.

With the same approximations as before, which are valid for cylinders with less than 10^{12} electrons/cm length, we find for c_n in the same way as eq. (64) was found

$$c_n \approx \left(\frac{2}{k\,a}\right)^{2\,n}\,\frac{n\,!\,(n-1)\,!}{\pi}\,\frac{1 + \varepsilon + i\,\varepsilon\,n\,\pi/\varepsilon'\,a}{1 - \varepsilon}. \tag{72}$$

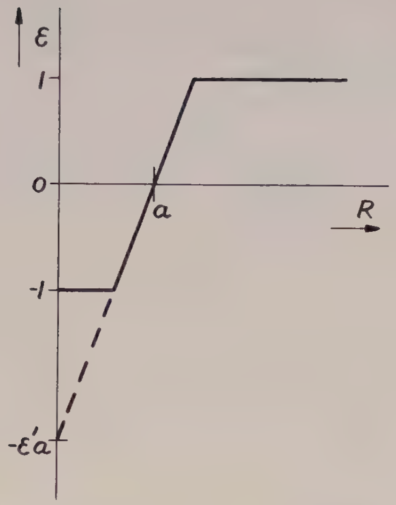

Fig. 11. Dielectric constant as function of R for a homogeneous cylinder with a diffuse boundary layer. Physical significance of $\varepsilon' a$.

The point representing c_n in the complex plane is now moving along a curve below the real axis, when ε is negative. This means that as compared with the case of a sharp cylinder, c_n is now always further away from the point i, and the resonance peak becomes lower and broader. Numerical experience shows that the curve followed by c_n is fairly straight, and the point of closest approach to i is not very far from the imaginary axis. To estimate the influence of the inhomogeneity in the cylinder boundaries we can therefore put $\varepsilon = -1$ in eq. (72), and the corresponding distance from c_n to i is

$$1 + \left(\frac{2}{ka}\right)^{2n} \frac{n!\,(n-1)!}{\pi} \frac{1}{2} \frac{n\pi}{\varepsilon' a}.$$

This is approximately equal to

$$\left(\frac{2}{ka}\right)^{2n} \frac{(n!)^2}{2\varepsilon' a} \tag{73}$$

if ka is small, and ε' small enough to give a noticeable effect.

With a variation of the dielectric constant as shown in Fig. 11, $\varepsilon' a$ is the numerical value of ε which would have been found at the centre of the column if the gradient of ε in the boundary layer had been continued right into the middle of the column. When $\varepsilon' a$ becomes only a few times the actual centre value of ε, the boundaries may be called appreciably diffuse. The expression (73) then shows that the diffuse boundaries have reduced the resonance value of the echo amplitude by a factor,

$$\left(\frac{ka}{2}\right)^{2n} \frac{2\varepsilon' a}{(n!)^2} = \left(\frac{\alpha\varrho}{2}\right)^{n} \frac{2\varepsilon' a}{(n!)^2}$$

and for the peaks for which $n > 1$, this will usually rule out the resonance completely. The peak corresponding to $n = 1$, which is least affected, will have its wave amplitude reduced by a factor

$$\alpha \varrho \varepsilon' a$$

and the energy scattering diameter at resonance will be

$$D \frac{2\pi}{\lambda} = (4 \varrho \alpha \varepsilon' a)^2. \tag{74}$$

It is interesting to compare eq. (74) with BORN's approximation eq. (16) for parallel scattering. The comparison shows that as long as $\varepsilon' a \gg 1$ for the diffuse cylinder, the resonance peak in sagittal scattering will exist and give a stronger echo than for parallel scattering. $\varepsilon' a \gg 1$ corresponds to a fairly homogeneous cylinder with marked boundaries. But if $\varepsilon' a \approx 1$, the resonance peak will have been eliminated, and the scattering diameter for parallel and sagittal scattering are comparable even for negative dielectric constants. $\varepsilon' a = 1$ corresponds to a linear variation of the dielectric constant with distance from the axis, and a cylinder with such an electron distribution would be called completely diffuse. In the extreme case, when $\varepsilon' a = 1$ the validity of the approximations behind eq. (74) becomes questionable, but the variations in D, when $\varepsilon' a > 1$ and the approximations are valid, clearly show the trend towards a disappearing resonance peak for a cylinder whose boundaries become more and more diffuse.

The preceding discussion tends to exaggerate the reduction in the resonance peak. In particular, the point in the complex plane which represents c_n will not move along a curve with a tangent parallel to the real axis when it passes the negative imaginary axis. The shortest distance to the point i will therefore be somewhat smaller than the value given in eq. (73), and the corresponding resonance peak somewhat higher than stated in the eq. (74).

Finally, the effect of collisions should be estimated. In the formula (63) for S_n we can introduce the complex dielectric constant given in eq. (51), and hence take into account a small collision frequency ν_c in the case of a sharp cylinder. We are particularly interested in the case where the electron density is such that ε is near -1, i.e. $N e^2/\pi m \nu^2 = 2$. For small collision frequencies we then obtain

$$S_n \approx i n \nu_c / \nu \pi k a. \tag{75}$$

Eq. (75) can be compared with the corresponding expression (71) for S_n, in which collisions are neglected, but the effect of diffuse boundaries is included. For $\varepsilon \approx -1$, we find in this case

$$S_n \approx i n^2 \pi / k \varepsilon' a^2. \tag{76}$$

It is the imaginary terms in the eqs. (75) and (76) which remove the point c_n from the origin and reduce the resonance peaks. Collisions and diffuse boundaries would be about equally effective in this respect if

$$i n \nu_c / \nu \pi k a \approx i n^2 \pi / k \varepsilon' a^2$$

or

$$\nu_c / \nu \approx n \pi^2 / \varepsilon' a \approx 9 n / \varepsilon' a. \tag{77}$$

Bearing in mind that in ionospheric applications ν_c is usually much smaller than ν and that $\varepsilon' a$ can be expected to be between 1 and 10, it is clear from eq. (77) that the effect of collisions can never be comparable with the effect of diffuse boundaries. In a laboratory discharge one may produce electron distributions which are so sharply limited that the product $\varepsilon' a$ in the denominator on the right hand side of eq. (77) makes the boundary effect smaller than the collision damping, but this is not possible in the free atmosphere.

It is also worth noting that the diffuse boundaries become increasingly important for the resonance amplitude when the order number n increases, and this confirms the earlier conclusion that unless the boundaries are extremely sharp, the resonances for which $n > 1$ will be ruled out completely.

The discussion in this section has shown that for narrow and dense ionised columns two values of the dielectric constant, $\varepsilon = 0$ and $\varepsilon = -1$, are particularly significant. It was clear from the outset that complications might arise when ε goes through 0, owing to the extra singularity in the fundamental eq. (6). It might be more unexpected that $\varepsilon = -1$ is important. The discussion shows that $\varepsilon \approx -1$ is associated with a resonance in thin columns, and that the singularity which comes in at $\varepsilon = 0$ tends to destroy the resonance.

Plasma resonance in ionised columns

The formal analysis of the preceding section lead to a geometrical picture, in which the scattering amplitude is determined chiefly by the regular motion of the representative points c_n along the real axis or along a regular curve in the complex plane. On the other hand, the physical properties of the scattering process are slightly obscured by the more accidental complications due to the introduction of cylinder coordinates. In the present section a less formal interpretation will be given, which may bring out the physical ideas more clearly. In particular, the unexpected significance of the value $\varepsilon = -1$ will be more evident right from the beginning than in the mathematical analysis.

In the absence of a magnetic field two characteristic frequencies are associated with a homogeneous ionised gas, the critical frequency and the plasma frequency.

The critical frequency ν_{crit} is defined by $\varepsilon = 0$, or

$$1 - N e^2 / \pi m \nu_{\text{crit}}^2 = 0. \tag{78}$$

MAXWELL-LORENTZ' equations (3) show that the critical frequency is the lowest frequency for which $\varepsilon \geqq 0$ and plane waves can be propagated. For frequencies below ν_{crit} plane fields are exponentially attenuated, and the radiation can not progress through the ionised gas. It follows from this property that at normal incidence to the boundary of a deep homogeneous layer, the critical frequency separates the waves of lower frequency which are totally reflected, from the waves of higher frequency which penetrate the layer. In this way the critical frequency is of fundamental importance for the regular radio reflections from the ionosphere.

The plasma frequency is a different concept. Consider a uniform layer of ionised gas with the same number N of electrons and ions per unit volume

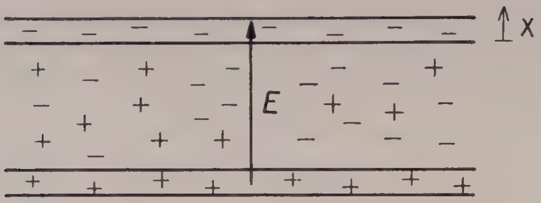

Fig. 12. A uniform polarisation of a homogeneously ionised layer gives rise to a uniform electric field inside the layer.

(Fig. 12). If all electrons are displaced a short distance x along the normal to the sides, a surface charge $\pm N e x$ appears on the surfaces of the layer, and a homogeneous electric field $E = 4\pi N e x$ tends to restore the equilibrium. The force $4\pi N e^2 x$ on each electron is proportional to the displacement, and the restoring force gives rise to harmonic oscillations with a frequency ν_{pl}, which is called the plasma frequency.

The plasma frequency is determined by the equation of motion

$$m\ddot{x} = -4\pi N e^2 x$$

which gives

$$m(2\pi\nu_{pl})^2 = 4\pi N e^2,$$

or

$$1 - N e^2/\pi m \nu_{pl}^2 = 0. \tag{79}$$

On the left hand side of eq. (79) we recognise the expression for the dielectric constant, and a comparison with eq. (78) shows that the critical frequency and the plasma frequency are identical. This is somewhat accidental, however, as it will be shown that the plasma frequency depends on the shape of the oscillating system as well as on the charge density.

As similar plasma frequency for an ionised cylinder has been derived by TONKS (1931). A small parallel displacement x of all electrons in a direction normal to the cylinder axis gives rise to a surface charge with density $N e x \cos\varphi$, where φ is the angle round the cylinder axis reckoned from the direction of x. The surface charge creates a homogeneous electric field inside the cylinder (Fig. 13). The internal field is equal to $-2\pi N e \nabla(R\cos\varphi)$, and the external field is $-2\pi N e \nabla(a^2\cos\varphi/R)$ where a is the cylinder radius and R the distance from the axis. Again the restoring force on the electrons is harmonic, and the equation of motion $m\ddot{x} = -2\pi N e^2 x$ determines the plasma frequency ν_{pl} for a cylinder. A comparison with the calculation for the plane case shows that the force on each electron is now reduced to one half of the previous value owing to the change in the geometrical shape, and the plasma frequency is determined by

$$1 - N e^2/\pi m \nu_{pl}^2 = -1. \tag{80}$$

According to eq. (80) a homogeneously ionised cylinder can oscillate in a free sagittal mode, whose frequency adjusts itself so that the dielectric constant becomes equal to -1. It is at once apparent that the first resonance peak

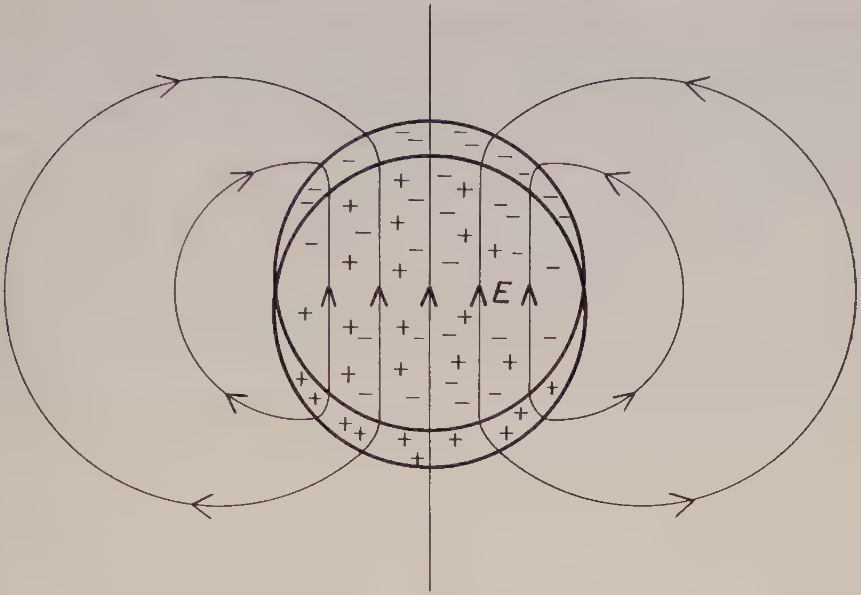

Fig. 13. A uniform polarisation of a homogeneously ionised cylinder gives rise to a uniform electric field inside the cylinder and a dipole field outside.

($n = 1$) in the sagittal scattering by a narrow cylinder is due to coupling between this mode and the incident wave.

A similar calculation for the case of a homogeneous sphere shows that the plasma frequency for a sphere is determined by the condition $\varepsilon = -2$. The plasma frequency of a cylinder was lower than that of a plane slab owing to the additional inertia associated with the electric field outside the cylinder, and the effect of the still greater external field energy in the case of a sphere is evident.

Oscillations of this kind can be excited in any homogeneously ionised region of ellipsoidal shape. They will be perfectly harmonic because a uniform polarisation of an ellipsoid induces a homogeneous restoring field inside the body, proportional to the displacement of the charges. The dielectric constant corresponding to the plasma frequency will in general be negative, and only tend towards zero when the ellipsoid degenerates into a plane slab.

The resonant response of a plasma has been verified experimentally by TONKS (1931), who inserted discharge tubes of various shapes in a condenser, and showed that for certain values of the discharge current a discharge tube becomes equivalent to a resonator located inside the condenser.

The qualitative discussion has so far led to the conclusion that an ionised cylinder has a natural frequency which is associated with a dipole oscillation in a direction normal to the cylinder axis. The next step is to study the excitation of the dipole oscillation by means of an incident radio wave, and calculate the scattering diameter at resonance.

According to eq. (65) the energy scattering diameter becomes comparable with the wave length when resonance occurs. This is a general property of resonators in many branches of physics, and as a simple illustration we can consider the scattering of radiation by an electric dipole (cf. BECKER, 1933).

An electromagnetic wave will set a harmonically bound electron into oscillations, and in the absence of any mechanical damping, the amplitude of the oscillations is limited only by the reradiated energy. The equation of motion for the electron will be

$$m\ddot{x} + (2e^2/3c^3)(2\pi\nu)^2\dot{x} + m(2\pi\nu_0)^2 x = eE\,exp\,(i2\pi\nu t) \tag{81}$$

where e and m are charge and mass of the electron, ν_0 is the frequency of the harmonic binding, and E and ν are amplitude and frequency of the incident radiation. The damping term is due to the radiation loss, and is a suitable approximation under stationary conditions.

At resonance, when $\nu = \nu_0$, we obtain from eq. (81) for the amplitude of the mechanical oscillation

$$A = 3c^3 eE/2e^2(2\pi\nu)^3. \tag{82}$$

To this amplitude corresponds a rate of radiation

$$\frac{2}{3}\frac{e^2}{c^3}A^2(2\pi\nu)^4 = \frac{3}{\pi}\lambda^2\left(\frac{cE^2}{8\pi}\right). \tag{83}$$

Since $(cE^2/8\pi)$ is the average flux in the incident radiation, eq. (83) shows that the scattering cross section at resonance of a harmonic oscillator is of the order of square wave length. A similar calculation for a two-dimensional harmonic oscillator would, of course, have shown that the scattering diameter at resonance is of the order of a wave length, which confirms in an elementary way the result of the mathematical discussion in the preceding section.

It is important to observe that the scattering diameter of a homogeneous cylinder at resonance depends only on the wave length of the incident radiation, and not on the dimensions or the detailed structure of the resonator. We note, for instance, that in eq. (65) neither the radius of the cylinder nor the number of electrons per unit length appear, and eq. (83) contains no reference to the charge or mass of the bound electron.

A superficial comparison of an ionised cylinder with a metal rod might have been seriously misleading on this point.

A metal cylinder can oscillate in a free dipole mode normal to the cylinder axis, and the frequency is such that the associated wave length *in vacuo* is comparable with the cylinder diameter. Hence, in the case of a metal cylinder, the following three lengths are comparable:

1) Cylinder diameter
2) Vacuum wave length of dipole oscillation
3) Scattering diameter at resonance for a thin cylinder.

The similarity between 2) and 3) is a general property, but 1) is comparable with 2) and 3) only for a metallic conductor and not for an ionised gas. The physical reason for the difference is that in a metal cylinder the energy is either electric or magnetic, and the kinetic energy of the charges is negligible, whereas in a thin ionised cylinder the energy is either electric or kinetic, and the magnetic energy is negligible. It is well known that a rough estimate of the natural frequency for a metal cylinder or a sphere can be found by equating the electric energy when the polarisation has a maximum, to the magnetic energy a quarter of a period later. This calculation is distinctly different from the derivation of the plasma frequencies in eqs. (79) and (80), where the equations of motions clearly illustrate the conversion of electrical energy into mechanical energy, and back again, for an ionised medium.

The resonant response of an ionised column and the difference between a metal and an ionised gas can be verified by laboratory experiments. A cylindrical discharge tube with a diameter which is a small fraction of a wave length, will behave as a very thin metal wire for parallel scattering, but at resonance in sagittal scattering it is equivalent to a metal cylinder with a diameter many times greater than the width of the tube. These predictions have been verified by ROMELL (1951), who has studied the scattering of 30 cm radio waves by a discharge tube with a diameter of 3 cm.

The experiments show a high resonance peak in sagittal scattering at a charge density which corresponds to a value of the dielectric constant near to -1. A comparison with the reflecting power of a metal strip indicates that the scattering diameter at resonance is comparable with, but somewhat smaller than, a wave length. No scattering at all could be detected for parallel scattering with the apparatus used. A quantitative explanation of two subsidiary resonance peaks in sagittal scattering is difficult at present, and more observations would be desirable, particularly of the variations in the scattering diameter with the angle round the cylinder axis.

The dipole oscillations are the simplest class of oscillations which can be excited in an ionised column or sphere, and they account for the resonance maximum corresponding to $n = 1$ in the series in eq. (38). In addition, an ionised cylinder can oscillate in more complicated sagittal modes, which account for the resonances of higher orders in eq. (38). To construct more general solutions we consider a homogeneous cylinder and assume that all electrons oscillate round their equilibrium positions with the same frequency ν, but not necessarily in parallel directions and with identical amplitudes as in the dipole oscillation. Calling the displacement of an electron $\mathbf{s}$, we try a solution

$$\mathbf{s} = \nabla V \, exp\,(i 2 \pi \nu t) \text{ where } \nabla^2 V = 0. \tag{84}$$

In the case of the dipole oscillation, the surfaces $V = \text{const.}$ were parallel planes, but in the more general solutions V shall only satisfy Laplace's equation.

According to eq. (84), div $\mathbf{s} = 0$, which means that inside the homogeneous cylinder no space charge is created during the oscillations, and charges appear only on the surface of the cylinder. The equation of motion

$$m \ddot{\mathbf{s}} = e \mathbf{E}$$

shows that

$$\mathbf{E} = -(m/e)(2 \pi \nu)^2 \mathbf{s}. \tag{85}$$

Since $\mathbf{E}$ is proportional to $\mathbf{s}$, div $\mathbf{E}$ will also vanish, which is consistent with the absence of space charge inside the cylinder.

The appropriate solutions of eq. (84) are, inside the cylinder

$$V = A R^n \cos n\varphi, \qquad n = 1, 2, 3 \ldots \tag{86}$$

and outside the cylinder

$$V = A a^{2n} \cos n\varphi / R^n \tag{87}$$

where a is the cylinder radius, and A a constant. The solutions show that the discontinuity in the normal component of the electric vector across the boundary is

$$2 (m/e) (2\pi\nu)^2 n A a^{n-1} \cos n\varphi. \tag{88}$$

The surface charge σ can be calculated from eq. (84), which gives, apart from the time factor

$$4\pi\sigma = 4\pi N e n A a^{n-1} \cos n\varphi. \tag{89}$$

Equating the expressions (88) and (89), we obtain as the relation between electron density and plasma frequency for the higher modes

$$2 (m/e) (2\pi\nu)^2 = 4\pi N e$$

or

$$\varepsilon = 1 - N e^2/\pi m \nu^2 = -1 \tag{90}$$

independently of n.

Each of the solutions (86) describes a free oscillation which corresponds to a term with the same value of n in eqs. (37) or (38) for the scattering cross section. In agreement with the discussion before eq. (65), eq. (90) shows that all modes have the same resonance frequency in this approximation, corresponding to $\varepsilon = -1$.

In Fig. 13 it was shown that in the dipole oscillation ($n = 1$) all electrons throughout the column oscillate in the same direction. The behaviour of the higher modes is markedly different in that only a shallow region near the surface takes part in the oscillation. As an illustration, Fig. 14 shows the electric field lines for the case $n = 2$, and for higher values of n, the oscillation will have still more the character of a surface phenomenon. If the surface of the cylinder is diffuse instead of perfectly sharp, the higher modes will be more seriously disturbed than the dipole mode which pervades the whole cylinder. This is in agreement with the conclusion of the preceding section that unless the boundaries are extremely sharp, the resonances for which $n > 1$ will be ruled out completely.

The solutions of eq. (84) exhaust the number of modes which must be considered for the sagittal scattering, because each term in the series (38) for the scattering diameter can be associated with one of the modes obtained from eq. (84). The restriction in eq. (84) that space charge is absent and hence div $\mathbf{E} = 0$ inside a homogeneous region does not exclude any significant solutions, as can be seen from the first of the eqs. (3). In eqs. (3) it is only assumed that a linear approximation is permissible in the equation of motion, and that all parts of the system oscillate with the same frequency. When ε is constant, the first eq. (3) gives div $\mathbf{E} = 0$, and this relation is therefore part of the theory right from the beginning, and is not an additional restriction on later equations such as eq. (84).

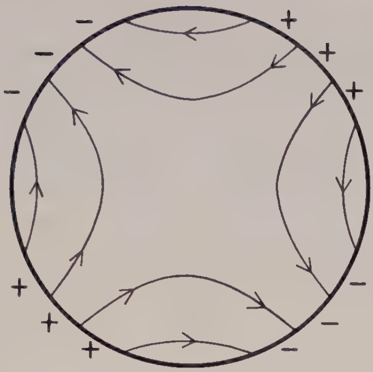

Fig. 14. Electric field lines and stream lines for the velocity in the quadripole oscillation of a homogeneously ionised cylinder.

At first sight it might be unexpected that the higher modes can be excited at all by an incident plane wave when the cylinder radius is small compared with the wave length, and still more that the scattering diameter at resonance is comparable with a wave length even for the higher modes according to eq. (65). Qualitatively, this can be understood, however, by studying a simplified model of the higher modes similar to the dipole discussed in the eqs. (81)–(83). The free oscillation shown in Fig. 14 for $n = 2$ can be represented by two dipoles separated by a short distance a and compelled to oscillate in opposite directions. The driving force in the right hand side of eq. (81) would then be reduced by a factor $(2\pi a/\lambda)$ in the case of this quadripole, but the radiation field would also be reduced by the same factor if the mechanical amplitude were the same. Therefore, the damping term in eq. (81) would be reduced by $(2\pi a/\lambda)^2$, and the amplitude A in eq. (82) would be greater by a factor $(\lambda/2\pi a)$. Squaring the amplitude, and remembering the factor $(2\pi a/\lambda)^2$ reducing the radiation resistance, we obtain finally that the rate of radiation is comparable with the expression (83) for the dipole, and the scattering cross section is much the same.

This is in agreement with the conclusion in eq. (65).

We note that for the quadripole, and still more for the higher modes, the amplitude at resonance must be increased in order to compensate for the reduced efficiency of the radiator as compared with the dipole. Therefore, it will take a longer time for the higher modes to build up to the full resonance amplitude, and interruptions due to collisions are more serious than for the dipole mode. This is in accordance with the conclusions drawn from eq. (75), which show that collision damping is more serious for modes of higher order than for modes of low order.

Plasma resonance in meteor trails

The effect of plasma resonance on the scattering of radio waves has now been established by calculations and by qualitative reasoning. Significant predictions have been verified experimentally by Romell.

Similar effects can occur in radio echoes from the upper atmosphere if suitable scattering objects are available, but the observations will require special apparatus built for this purpose. In the absence of observations the discussion will necessarily be tentative, but it should be possible to decide beforehand whether an experiment is likely to be of interest and give positive results.

It is natural to try observations on meteor trails at first, because we know more about meteor trails than about any other small or narrow structures in the ionosphere. Initially, the trails are straight columns of ionisation, so plasma resonance can be observed as a difference in the strength of parallel and sagittal echoes. The moment immediately after formation of the trail may be favourable, because the electron density is then greatest, and distortions due to turbulence are not yet developed.

The discussion of the ionised trails produced by meteors indicated that the trails are initially formed as narrow columns with diameters of the order of centimetres or decimetres. If this is correct, it is advisable to carry out the observations on wave lengths longer than, say, 2 metres in order to observe the plasma resonance in a clear form, and avoid complications due to poor convergence of the series (38) for the energy scattering diameter. The discussion of the eqs. (60) and (64) shows that convergence difficulties set in when $2\pi a/\lambda$ becomes comparable to unity. Physically, they are associated with the gradual transition to diffraction optics and geometrical optics for a reflecting cylinder much wider than a wave length, when the series (38) is becoming useless.

To give more precise limits for the useful range of wave lengths, we consider the number of electrons in typical meteor trails. According to eq. (56), the dielectric constant of a homogeneous cylinder is approximately

$$\varepsilon = 1 - 10^{-12}\,\alpha/(k\,a)^2$$

where α is the number of electrons per cm path of the meteor trail. Resonance will occur when $\varepsilon = -1$, i.e. when

$$(k\,a)^2 \approx \tfrac{1}{2}\,10^{-12}\,\alpha. \tag{91}$$

In the discussion of the ionised trails produced by meteors it was shown that we can expect a fair number of trails with $\alpha = 10^{10}$ or 10^{11} electrons/cm. Corresponding to these values, eq. (91) shows that the wave length should be chosen so that $k\,a > 0.07$ or $k\,a > 0.22$ respectively, which is equivalent to $\lambda > 100\,a$ and $\lambda > 30\,a$.

Bearing in mind that we expect a to be of the order of centimetres or decimetres, these figures suggest that on a wave length of 10 metres, a great proportion of the trails due to visual meteors will show plasma resonance. On a wave length of 4 metres, we are more likely to be in a transition region where some trails will show resonance, and some not.

It is easier to build a suitable aerial system for 4 metres wave length, because the dimensions are smaller, so it may be preferable to start the observations on this wave length. If many echoes are observed as polarised on 4 metres, there will be still more of the polarised echoes on 10 metres. On the other hand, if the polarisation experiment on 4 metres gives a negative result, a repetition of the experiment on a longer wave length should be considered, because the expression for the dielectric constant shows that with increasing wave length, resonance will occur sooner or later.

The polarisation experiment will serve as a check on the theory of meteor ionisation. The estimates given in the beginning of the present paper, and in more detail in an earlier review (Herlofson, 1948), are thought to be based on sound principles. But it should be borne in mind that an accurate analysis of the evaporation process is difficult. Moreover, other authors have come to very different conclusions about dimensions and electron densities (references, see Herlofson, 1948; Millman & McKinley, 1949).

The expected polarisation depends on the estimates of diameter and number of electrons per unit length in the trails, and on the results of the scattering theory. If the diameter of the trails is tens of metres or more, as suggested by some authors, the polarisation will either be erratic, or entirely absent. It will be erratic, both in direction and magnitude, if the electron production is so great that even a trail many wave lengths in diameter behaves as a perfect reflector, because the simple scattering theory for a thin trail is then replaced by a complicated transition to diffraction optics. The polarisation will be absent if the electron production is so small that the trail is nearly transparent, and Thomson scattering by each electron is predominant.

The occurrence of sagittal echoes which are always stronger than the parallel echoes for wave lengths of 4 to 10 metres, or longer, will therefore give direct information about the trail diameter, which must then be small, and about the electron density which must be great enough to reduce the dielectric constant to -1.

With simultaneous observations of the parallel and sagittal echo it will also be possible to obtain more precise information about the electron production than before. In the observations up till now (Lovell & Clegg, 1948), the number of electrons per cm length of the trails has been deduced by a formula which is equivalent to eq. (16), and is based on the assumptions that the column is narrow compared with a wave length, and each electron scatters as a free electric charge. It would clearly be desirable to have a direct proof that the assumptions are justified in a particular case. This can be obtained by observing the *parallel* echo at the moment when the *sagittal* echo shows maximum resonance (cf. Fig. 10). The amplitude of the parallel echo will then give the correct number of electrons by Blackett and Lovell's formula, provided the number deduced is well below 10^{12} electrons/cm length. At the same time the condition $\varepsilon = -1$ at resonance will give a precise value for the effective radius of the column. The reason is that according to the eqs. (56) and (57), resonance can only occur in the sagittal echo when $ka \ll 1$, if $\alpha \ll 10^{12}$ electrons/cm. But under these conditions, eq. (21) shows that Born's approximation is valid for *parallel* scattering, and furthermore the discussion following eq. (18) shows that phase differences over the cross section can be neglected, which are exactly the assumptions underlying Blackett and Lovell's theory.

At present there are no direct observations to support the view that plasma resonance occurs in meteor trails. Some indirect observations, however, may be taken as favourable evidence. With apparatus of equivalent sensitivity a greater number of echoes are observed on a wave length of 8 metres than on 4 metres (Lovell, 1948, pp. 428–29), an effect which has not yet been properly explained. The reason may be that plasma resonance can occur more readily on the longer wave lengths, and faint meteors may produce meteor trails with an electron density sufficient to produce resonance on 8 metres, but not on 4 metres. With random orientation of the trails, a number of sagittal echoes would then be observed on 8 metres, but would not appear on 4 metres. The explanation is hypothetical, but polarisation measurements on the additional 8 metre echoes would be a decisive test.

Plasma resonance in ionospheric scattering centres

There is evidence that the ionosphere contains numerous irregularities which contribute to the fading of radio waves (RATCLIFFE, 1948; MITRA, 1949; McNICOL, 1949), and as the observations are carried out on longer wave lengths than the meteor observations, it is natural to consider a possible influence of plasma resonance.

Several factors contribute to fading in short wave transmission. Interference between a ground wave and a sky wave is one source of fading, which can be eliminated by suitable directional aerials, sensitive only to the sky wave. Another source is interference between the ordinary and extraordinary waves, which travel along different paths in the ionosphere. This type of fading can be eliminated by the use of polarising aerials, which respond only to one of the two waves. By such precautions it is possible to isolate a single ray connecting transmitter and receiver via the ionosphere, and one might expect that fading would then be absent.

This is, however, not the case, and it has been shown by RATCLIFFE and his collaborators that this single ray fading can be interpreted as the effect of numerous small scattering centres, which introduce fluctuations into the steady echo from the main ionospheric layers. The scattering centres drift with the winds and the turbulent motions in the ionosphere, they are created and disappear, so the effect on the radio echoes will have the character of a random fluctuation.

By extensive statistical analysis of the fluctuations (cf. BOOKER, RATCLIFFE & SHINN, 1950) it has been possible to extract values of the wind velocities in the ionosphere, and the results agree with observations of the drift of luminous night clouds (STÖRMER, 1935) and persistent meteor trails (e.g. STÖRMER, 1937). In the statistical analysis it is sufficient to describe the scattering centres as small irregularities located in or below the ionospheric layers without specifying their properties more precisely, and very little is known otherwise about their exact location or physical constitution.

If it is taken for granted that plasma resonance will eventually be observed in meteor trails, the effect should always be borne in mind in the discussion of scattering by any kind of small irregularities in the ionosphere. For an appreciable effect on the scattering cross section, the dimension of the scattering region along the electric vector must be a small fraction of a wave length, the electric vector of the incident wave must point across the surfaces of equal electron density, and the boundaries of the ionised region must not be too diffuse. The dielectric constant must be negative. The shape of the scattering centres is not particularly important, and will only determine the value of the dielectric constant for which resonance takes place.

These general principles were derived on the assumption that the ionised region was surrounded by vacuum. The scattering centres in the ionosphere may, however, be embedded in a medium where the dielectric constant differs from unity, which modifies the condition for plasma resonance. An inspection of the scattering theory shows that the significant quantity is the *ratio* between the dielectric constants of the ionised region and the surrounding medium. This would lead to the conclusion that resonance in a homogeneous cylinder embedded in a uniform medium with dielectric constant ε_1 (> 0) takes place when the dielectric constant of the cylinder is $-\varepsilon_1$. In particular, if the dielectric constant of the surrounding medium is small and positive, it is sufficient that the dielectric constant of a cylinder at resonance is small and negative. Similarly, for a sphere the ratio between the internal and the external dielectric constants should be -2 for resonance to occur.

Without much calculation these results can also be derived from the qualitative study of the free oscillations of cylinders or spheres.

For the dipole oscillation of a cylinder, the internal electric field E is $A \nabla (R \cos \varphi)$ and the external field $A \nabla (a^2 \cos \varphi / R)$, where A is a constant, a is the cylinder radius, and R, φ are cylinder coordinates. If all electrons oscillate with the same frequency ν, the amplitude of the electron oscillations is $eE/m(2\pi\nu)^2$. When the cylinder is embedded in an ionised gas, both the plasma inside and outside will contribute to the surface charge on the boundary (cf. Fig. 13), and the boundary condition for the normal component of the electric field will now be

$$4\pi (N_1 e + N_2 e)\, eA/m(2\pi\nu)^2 = 2A$$

where N_1 and N_2 are the internal and external electron densities.

A slight rearrangement gives

$$1 - N_1 e^2/\pi m \nu^2 = -(1 - N_2 e^2/\pi m \nu^2). \tag{92}$$

Eq. (92) shows again that resonance in a cylinder occurs when the internal and external dielectric constants are equal in magnitude but opposite in sign. The additional electrons outside the cylinder have increased the plasma frequency, because the restoring force due to the surface charge is now greater than for the cylinder *in vacuo*. The result is that the plasma frequency for a cylinder embedded in an ionised region comes nearer to the frequency for a plane slab.

Applying this result to echoes from the ionospheric layers, it is clear that a thin layer round the level where the dielectric constant vanishes will be particularly favourable for plasma resonance (Fig. 15). In order to discuss a simple model, we consider the reflection of radio waves at a point vertically above a transmitter. Below the level where $\varepsilon = 0$, very little energy is reflected because the refractive index varies only little in a distance equal to a wave length, and the whole incident energy is carried right up to the reflection level. If vertical mixing of the air takes place at these heights, the ionosphere slightly below the level where $\varepsilon = 0$ will contain elements which have been transported downwards from positions slightly above the level $\varepsilon = 0$.

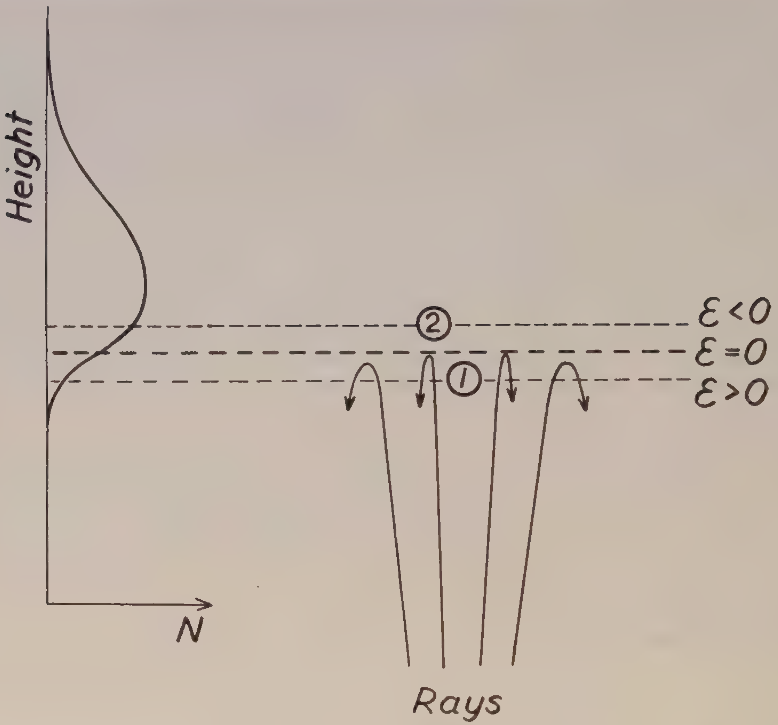

Fig. 15. To the left is shown the electron density N as function of height in a typical ionospheric layer. To the right are shown rays of an incident wave which are reflected at the level where $\varepsilon = 0$. Vertical mixing will bring ionised gas from the level 2 where $\varepsilon < 0$ down to the level 1 where $\varepsilon > 0$, and vice versa, and the radio wave will be scattered by the irregularities.

These elements will act as irregularities and scatter the incident radio wave. The interesting point is now that by this mechanism an irregularity with a small *negative* dielectric constant is temporarily embedded in a region with a small *positive* dielectric constant, and can therefore be a very efficient scatterer owing to plasma resonance, although none of the two dielectric constants need be much different from 0. Since small variations of ε round the value 0 are sufficient, the process will be active right up to the critical frequency for the ionised layer.

In the absence of disturbances the surface where $\varepsilon = 0$ would have acted as a smooth mirror. But the present discussion indicates that since the mirror is formed by an electron gas, any small imperfection in the surface appears as a serious irregularity in the radio echo. The fluctuation in the echo will be further enhanced by the optical properties of the ionosphere below the scattering centre (Fig. 16). Some distance away from the irregularity the scattered field will be more or less an expanding spherical wave, but the radiation will be bent towards the vertical before it escapes from the lower fringe of the ionised layer.

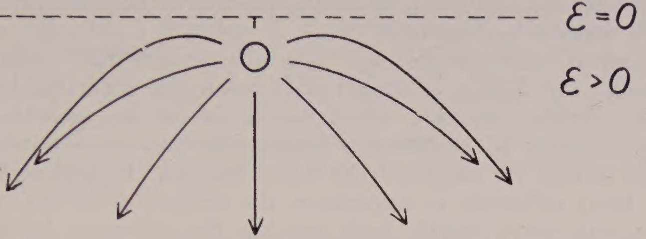

Fig. 16. Rays from an isotropic radiator inside an ionospheric layer are bent downwards.

The question whether vertical mixing of the air does take place in the ionosphere requires special consideration.

The density of the air in the lower ionosphere differs from the density in the troposphere by six powers of ten or more, so the molecular properties of the air are very different at ground level and above 100 kilometres. But this is not of importance for the macroscopic behaviour of the atmosphere, which is remarkably similar in the troposphere and in the *E*-layer.

Under normal conditions the wind speed in the troposphere is small near the ground, but increases to a maximum of some tens of metres per second at heights of about 10 kilometres. In the lower parts of the ionosphere visual observations of persistent meteor trails (e.g. STÖRMER, 1937) show that the trails are distorted after a few minutes due to the variations of the wind speed with height, and typical values of the variations are some tens of metres/second over height intervals of 10 kilometres. This is much the same value as found in the troposphere. Motion on a smaller scale is evident in photographs of luminous night clouds (STÖRMER, 1935), which occur at heights of 80—85 kilometres. The luminous night clouds show regular waves and finer filaments of dimensions and general appearance very similar to those of thin stratus clouds in the troposphere. Thirdly, the persistent meteor trails are gradually dispersed at a rate comparable with the rate of dispersal for smoke columns in the troposphere, which indicates that even eddy diffusion is comparable at the two levels.

This evidence is not really conclusive, but as far as it goes it supports the view that the dynamical behaviour of the atmosphere is fairly similar in the troposphere and in the lower ionosphere.

In the troposphere vertical motion of the air can be caused by irregularities of the ground, by instability due to the condensation of water vapour, and by strong variations of the wind velocity with height. The first two causes are absent in the ionosphere, but judging from the meteorological data discussed above, it is likely that vertical mixing due to wind variations with height will be about as strong as in the troposphere. We might then expect exchange of elements in the air to take place over distances of some tens or hundreds of metres in the vertical direction.

The preceding discussion shows that both the electrodynamical and the meteorological conditions in the ionosphere are such that single ray fading at vertical incidence can be caused by processes close to the level of total reflection. A precise statement on the effects to be expected at oblique incidence is difficult, as the behaviour of scattering elements located at the caustic of an incident wave field is entirely unknown.

On the present observations one can not exclude the alternative point of view that single ray fading is caused well below the reflection level by fluctuations of the electron density, which would distort the smooth wave surfaces of the echo. It may be possible to distinguish between the possibilities by a study of the fading on neighbouring wave lengths. If plasma resonance near the level of total reflection is important, the properties of the fading will vary very rapidly with wave length, both because the reflection level will move to a new position, and the resonance mechanism in itself is fairly selective. One might also be able to identify levels where vertical mixing is weak or absent, if only a thin layer contributes to the single ray fading.

It is inconceivable that single ray fading can be due to scattering by meteor trails.

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Note added in proof: In a private communication Dr. J. A. Clegg of the University of Manchester has informed the writer that he has recently detected the polarisation of meteor echoes in a number of cases on a wave length of 4 metres, and that the results are in agreement with the properties of plasma resonance. Dr. Clegg's observations would change the tentative predictions about polarisation in meteor echoes into a discussion of an established effect, and give additional reason to search for plasma resonance in radio echoes from other scattering centres in the ionosphere.

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